EFFECT ALGEBRAS AND ω-CATEGORIES

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https://arxiv.org/abs/2303.17257

Abstract

We show how an effect algebra E can be regarded as a category, where the morphisms $x \to y$ are the elements f such that $x \leq f \leq y$. This gives an embedding $\mathbf{EA} \to \mathbf{Cat}$. The interval $[x, y]$ proves to be an effect algebra in its own right, so E is an $\mathbf{EA}$-enriched category. The construction can therefore be repeated, meaning that every effect algebra can be identified with a strict ω-category.

1. Effect Algebras

An *effect* or 'fuzzy observable' on a Hilbert space is a self-adjoint operator whose spectrum lies between 0 and 1. An *effect algebra* [1] is an axiomatization of the effects on a Hilbert space. It generalizes structures from logic, probability theory and quantum theory [2].

Definition A *partial commutative monoid* A is a set A with a *partial* operation $\oplus : A^2 \rightharpoonup A$ and an element 0 such that:

$$x \oplus y \simeq y \oplus x \tag{1}$$
$$x \oplus (y \oplus z) \simeq (x \oplus y) \oplus z \tag{2}$$
$$x \oplus 0 = x \tag{3}$$

(Here $E_1 \simeq E_2$ means E_1 is defined iff E_2 is defined, in which case they are equal.)

Definition An *effect algebra* is a partial commutative monoid E with a (total) operation ${}^\perp : E \to E$ such that:

- $x^\perp$ is the unique element such that $x \oplus x^\perp = 0^\perp$
- $x \oplus 0^\perp$ is defined if and only if $x = 0$

Examples The set of *effects* on a Hilbert space; any Boolean algebra; the discrete probability distributions on a set; the measurable functions on a measurable space

2. Effect Algebras as Categories

Monoids are categories with one object, which suggests that partial commutative monoids should be something category-like, with $\oplus$ as composition. Following this idea gives us:

Definition Given an effect algebra E, define a category $\mathbb{B}E$ with:

- objects $|\mathbb{B}E| = E$
- $\mathbf{Hom}[x, y] = [x, y]^1 \cong [0, y \ominus x]$
- composition is $\oplus : [y, z] \times [x, y] \to [x, z]$

This defines a functor

$$\mathbb{B} : \mathbf{EA} \to \mathbf{Cat}\ .$$

But more is true: the intervals $[x, y]$ are themselves effect algebras, and $\mathbb{B}E$ is an $\mathbf{EA}_*$-enriched category[2]. This means the process can be iterated...

[1] Here $[a, b]$ is the interval $\{x \in E \mid a \leq x \leq b\}$.
[2] Here $\mathbf{EA}_*$ is the category whose objects are the effect algebras together with $\emptyset$, and whose morphisms are homomorphisms of effect algebras and empty functions.

3. Effect Algebras as (strict) 2-Categories

The (non-empty) hom-sets $\mathbf{Hom}[x, y]$ in the category associated with an effect algebra E are effect algebras in their own right. Composition is easily seen to be a morphism of effect algebras (or the empty function). Hence we can replace the hom-effect algebras with the associated categories, and since $\mathbb{B}$ is monoidal, composition turns into a bifunctor:

$$\mathbb{B}\circ : \mathbb{B}(\mathbf{Hom}[y, z] \times \mathbf{Hom}[x, y]) \to \mathbb{B}\mathbf{Hom}[x, z]$$

This means that replacing the hom-effect-algebras (or empty hom-sets) with the corresponding categories (or with the empty category) yields a strict 2-category, in which both horizontal and vertical composition are built from $\oplus$. This construction can be extended to a functor into the category of 2-categories:

$$\mathbb{B}^2 : \mathbf{EA} \to 2\text{-}\mathbf{Cat}$$

We can also package all this by considering the aforementioned category $\mathbf{EA}_*$: the monoidal functor $\mathbb{B}$ can be extended to a functor that factors through $\mathbf{EA}_*$-enriched categories

$$\mathbb{B} : \mathbf{EA}_* \xrightarrow{\mathbb{B}^e} \mathbf{EA}_*\text{-}\mathbf{Cat} \hookrightarrow \mathbf{Cat}$$

and we can consider the change-of-base functor $\mathbb{B}_*$ that it induces: composing the two yields the desired functor

$$\mathbb{B}^2 : \mathbf{EA}_* \xrightarrow{\mathbb{B}^e} \mathbf{EA}_*\text{-}\mathbf{Cat} \xrightarrow{\mathbb{B}_*} \mathbf{Cat}\text{-}\mathbf{Cat} = 2\text{-}\mathbf{Cat}$$

4. Effect Algebras as ω-Categories

As for the case with categories stemming from effect algebras, the (non-empty) sets of 2-cells in the 2-category associated with an effect algebra also admits the structure of an effect algebra. It is also the case that horizontal and vertical compositions are morphisms of effect algebras. This means the functor $\mathbb{B}^2$ can be factored as well:

$$\mathbb{B}^2 : \mathbf{EA}_* \xrightarrow{\mathbb{B}^e} \mathbf{EA}_*\text{-}\mathbf{Cat} \xrightarrow{\mathbb{B}^e_*} (\mathbf{EA}_*\text{-}\mathbf{Cat})\text{-}\mathbf{Cat} \hookrightarrow 2\text{-}\mathbf{Cat}$$

It's now clear that we could carry on applying the construction inductively.
We find it more illuminating to directly address the limiting case instead: the strict ω-category $\mathbb{B}^\infty E$. Given an effect algebra E there's a reflexive globular set $\mathbb{B}^\infty E$

$$(\mathbb{B}^\infty E)_0 \overset{s_0,\ t_0}{\underset{i_0}{\leftrightarrows}} (\mathbb{B}^\infty E)_1 \overset{s_1,\ t_1}{\underset{i_1}{\leftrightarrows}} (\mathbb{B}^\infty E)_2 \xrightarrow{i_2} \dots$$

An n-cell is a sequence $[a_1, \dots, a_{2n+1}]$ such that $a_1 \leq \cdots \leq a_{2n+1}$.
The globular set can also be endowed with compositions

$$(\mathbb{B}^\infty E)_{n+k} \times_n (\mathbb{B}^\infty E)_{n+k} \to (\mathbb{B}^\infty E)_{n+k}$$

such that any choice of three increasing natural numbers $l < m < n$ gives us a 2-category with $(\mathbb{B}^\infty E)_l$ as its objects, $(\mathbb{B}^\infty E)_m$ as its 1-cells, $(\mathbb{B}^\infty E)_n$ as its 2-cells. This means interchange holds, hence we have a strict ω-category.

Discussion

This work describes an observation: an effect algebra naturally has the structure of an ω-category. We do not know any significant consequences of this fact yet, but we can speculate:

- The ∞-nerve of an ω-category is a simplicial set [3]. So we can regard an effect algebra as a simplicial set, and perhaps therefore a model of homotopy type theory.
- Jacobs [2] describes a general class of categories (the 'effectuses') the *predicates* on X (the maps $X \to 2$) form an effect algebra. What structure do these have as categories and ω-categories?

References

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