University of Gdańsk

International Centre for Theory of Quantum Technologies

Quantifying EPR: the resource theory of nonclassicality of common-cause assemblages

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ICTQT - University of Gdańsk

Quantum 7, 926 (2023) and arXiv:2209.10177

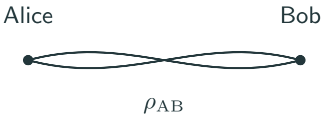
European Funds Smart Growth

Republic of Poland

FNP Foundation for Polish Science

European Union European Regional Development Fund

Entanglement

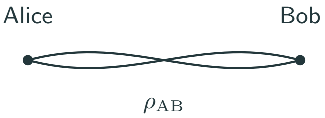


- Quantum teleportation
- Quantum Key Distribution

Nonlocality

- Device Independent QKD
- Randomness

Entanglement



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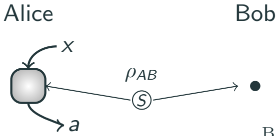
Einstein-Podolsky-Rosen (EPR) – a.k.a. steering

Traditional bipartite setup[1]



$$\sigma^{\mathrm{B}}_{a|x} = \mathrm{tr}_A \left\{ M_{a|x} \otimes \mathbb{I}_B \, \rho_{AB} \right\}$$

[1] H. Wiseman, S. J. Jones, A. C. Doherty. Phys. Rev. Lett. 98, 140402 (2007)

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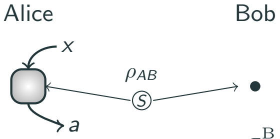

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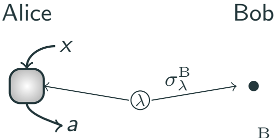
Classical common cause:



$$\sigma^{\mathrm{B}}_{a|x} = \sum_\lambda p(\lambda)\, p(a|x,\lambda)\rho^{\mathrm{B}}_{\lambda}$$

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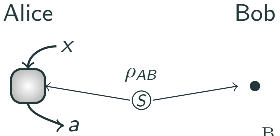

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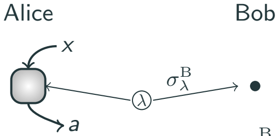

$$\sigma^{\mathrm{B}}_{a|x} = \sum_{\lambda} p(\lambda) \, p(a|x,\lambda) \rho^{\mathrm{B}}_{\lambda}$$

If $\sigma^{\mathrm{B}}_{a|x}$ does not admit of such classical model, then it is EPR non-classical

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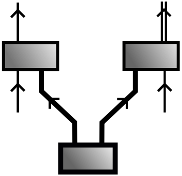
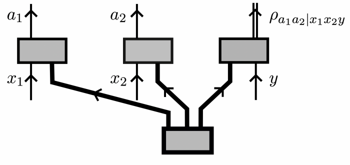
Generalised EPR scenarios

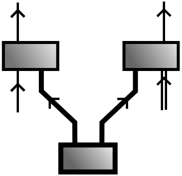
Bob-with-Input: States $\sigma_{a|xy}$



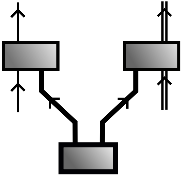
Multipartite Bob-with-Input:
States $\sigma_{a_1 a_2|x_1 x_2 y}$



Measurement-Device-Independent:
POVM elements $M_{ab|x}$



Channel 'steering':
Quantum channels $\mathcal{I}_{a|x}$



Assemblage: $\{\sigma_{a|xy}\}_{a,x,y}$, $\{\sigma_{a_1 a_2|x_1 x_2 y}\}_{a_1,a_2,x_1,x_2,y}$, $\{M_{ab|x}\}_{a,b,x}$, $\{\mathcal{I}_{a|x}\}_{a,x}$

Resource Theory I[2]

Scenario: traditional bipartite and multipartie EPR scenarios.

- Full set of available resources – *Enveloping theory*

  Common-cause form, quantum

- Criteria to assert what can be done freely

  Key feature: common cause among the wings

  free $\equiv$ classical

- Free set of resources and free set of transformations

  Processes whose common cause is classical

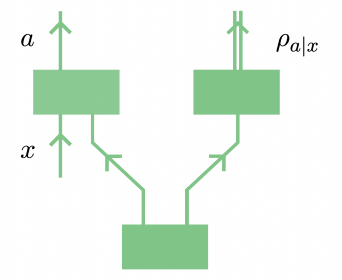
  Local Operations and Shared Randomness (LOSR)

[2] B. Coecke, T. Fritz, R. Spekkens. Information and Computation 250, 59 (2016)

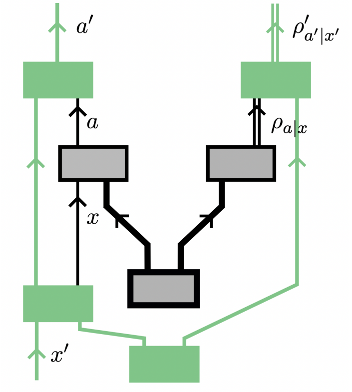
Free resources and transformations

Example: Traditional bipartite EPR scenarios

Free assemblage:



Free Transformation:



Resource Theory II

Scenario: Generalised scenarios – channel EPR scenarios

- Full set of available resources – *Enveloping theory*

  Common-cause form, from any theory $\longrightarrow$ non-signalling objects

  see Paulo Cavalcanti's talk

- Criteria to assert what can be done freely

  Key feature: common cause among the wings

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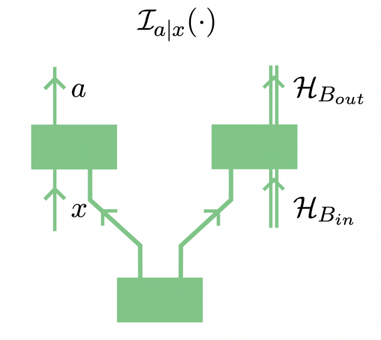
Free resources and transformations

Example: Channel EPR scenarios

Free assemblage:



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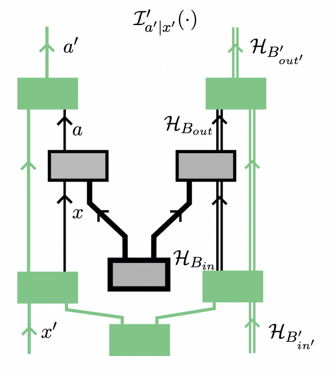

Results

- Result 1: Resource conversion under free operations in all scenarios can be evaluated with a single instance of a semidefinite program.

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Traditional scenarios: (quantum enveloping theory)

SDP 2. $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} \xrightarrow{LOSR} \mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$.
The assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ *can be converted into the assemblage* $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$ *under LOSR operations, denoted by* $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} \xrightarrow{LOSR} \mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$, *if and only if the following SDP is feasible:*

$$\begin{aligned} \textit{given} \quad & \{\{\sigma_{a|x}\}_a\}_x \,,\ \{\{\sigma'_{a'|x'}\}_{a'}\}_{x'} \,,\ \{D(a'|a,x',\lambda)\}_{\lambda,a',a,x'} \,,\ \{D(x|x',\lambda)\}_{\lambda,x,x'} \\ \textit{find} \quad & \{(W_\lambda)_{BB'}\}_\lambda \\ \textit{s.t.} \quad & \begin{cases} W_\lambda \geq 0 \,, \\ \mathrm{tr}_{B'}\{W_\lambda\} \propto \frac{1}{d}\,\mathbb{I}_B \quad \forall \lambda \,, \\ \sum_\lambda \mathrm{tr}_{B'}\{W_\lambda\} = \frac{1}{d}\,\mathbb{I}_B \,, \\ \sigma'_{a'|x'} = \sum_\lambda \sum_{a,x} D(a'|a,x',\lambda)\, D(x|x',\lambda)\, d_B\, \mathrm{tr}_B \left\{ W_\lambda \left(\mathbb{I}_{B'} \otimes \sigma^T_{a|x} \right) \right\} . \end{cases} \end{aligned} \tag{8}$$

When the conversion is not possible, we denote it by $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} \overset{LOSR}{\not\longrightarrow} \mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$.

Results

Generalised scenarios: (NS enveloping theory)

SDP 2. *The channel assemblage* $\boldsymbol{I}_{\mathbb{A}|\mathbb{X}}$ *can be converted to the channel assemblage* $\boldsymbol{I}'_{\mathbb{A}'|\mathbb{X}'}$ *under LOSR operations, denoted by* $\boldsymbol{I}_{\mathbb{A}|\mathbb{X}} \overset{LOSR}{\longrightarrow} \boldsymbol{I}'_{\mathbb{A}'|\mathbb{X}'}$*, if and only if the following SDP is feasible:*

$$
\begin{aligned}
\textit{given} \quad & \{J_{a|x}\}_{a,x}\,,\ \{J'_{a'|x'}\}_{a',x'}\,,\ \{D(a'|a,x',\lambda)\}_{\lambda,a',a,x'}\,,\ \{D(x|x',\lambda)\}_{\lambda,x,x'} \\
\textit{find} \quad & \{(J_{\xi\,\lambda})_{B_{in}B'_{in'},B_{out}B'_{out'}}\}_{\lambda}\,,\ \{(J_{F\,\lambda})_{B_{in}B'_{in'}}\}_{\lambda} \\
\textit{s.t.} \quad & \begin{cases}
J_{\xi\,\lambda} \geq 0 \quad \forall\lambda\,, \\
\mathrm{tr}_{B'_{out'}B_{in}}\{J_{\xi\,\lambda}\} \propto \mathbb{I}_{B_{out}B'_{in'}} \quad \forall\lambda\,, \\
\sum_\lambda \mathrm{tr}_{B'_{out'}B_{in}}\{J_{\xi\,\lambda}\} = \frac{1}{d_{B_{out}}d_{B'_{in'}}}\mathbb{I}_{B_{out}B'_{in'}}\,, \\
J_{F\,\lambda} \geq 0 \quad \forall\lambda\,, \\
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\mathrm{tr}_{B'_{out'}}\{J_{\xi\,\lambda}\} = J_{F\,\lambda} \otimes \frac{1}{d_{B_{out}}}\mathbb{I}_{B_{out}} \quad \forall\lambda\,, \\
J'_{a'|x'} = \sum_\lambda \sum_{a,x} D(a'|a,x',\lambda)\, D(x|x',\lambda)\, J_{a|x} * J_{\xi\,\lambda}\,.
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$$

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- Two-qubit system shared, state: $|\theta\rangle = \cos(\theta)\,|00\rangle + \sin(\theta)\,|11\rangle$.
- Alice's measurements: $x = 0 \rightarrow$ Pauli-Z, $\quad x = 1 \rightarrow$ Pauli-X

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- Assemblage elements: $\sigma^{\theta}_{a|x} = \mathrm{tr}_{\mathrm{A}} \left\{ M_{a|x} \otimes \mathbb{I}\, |\theta\rangle \langle\theta| \right\}$
- Family of assemblages: $\left\{ \Sigma^{\theta} \mid \theta \in (0, \frac{\pi}{4}] \right\}$

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Proof: family of monotones $\left\{M_{\eta} \mid \eta \in (0, \frac{\pi}{4}]\right\}$ such that:

$$\theta \neq \eta \quad \Rightarrow \quad M_{\eta}[\Sigma^{\theta}] < M_{\eta}[\Sigma^{\eta}]$$

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Hence: $M_{\theta} \Rightarrow \Sigma^{\eta} \not\rightarrow \Sigma^{\theta}, \quad M_{\eta} \Rightarrow \Sigma^{\theta} \not\rightarrow \Sigma^{\eta}$

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Bob-with-Input scenario: $\mathcal{H}_B =$ qubit, $\mathbb{X} = \{0, 1, 2\}$, $\mathbb{A} = \{0, 1\}$, $\mathbb{Y} = \{0, 1\}$

$$\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} = \left\{\sigma^{PTP}_{a|xy}\right\}_{a\in\mathbb{A},\, x\in\mathbb{X},\, y\in\mathbb{Y}} ,$$

$$\text{with} \quad \begin{cases} \sigma^{PTP}_{a|xy} = \xi_y \left\{\mathrm{tr}_{\mathrm{A}} \left\{\left(M_{a|x} \otimes \mathbb{I}_B\right) |\phi\rangle \langle\phi|\right\}\right\}, \\ M_{a|1} = \frac{\mathbb{I}+(-1)^a \sigma_x}{2}, \quad M_{a|2} = \frac{\mathbb{I}+(-1)^a \sigma_y}{2}, \quad M_{a|3} = \frac{\mathbb{I}+(-1)^a \sigma_z}{2}, \end{cases}$$

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$$\text{with} \quad \sigma^{PR}_{a|xy} = \begin{cases} |a \oplus xy\rangle \langle a \oplus xy| & \text{if } x \in \{0, 1\} \\ \frac{\mathbb{I}}{2} a & \text{if } x = 2. \end{cases}$$

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$$\Sigma^{PTP} \quad \not\longrightarrow \quad \Sigma^{PR} \quad \text{and} \quad \Sigma^{PTP} \quad \not\longleftarrow \quad \Sigma^{PR}$$

Comparison with other work

Traditional EPR scenario – multipartite:

Free assemblage: $\sigma_{a_1 a_2|x_1 x_2} = \sum_\lambda p(\lambda)\, p(a_1 a_2|x_1 x_2 \lambda)\, \rho_\lambda$

$p(a_1 a_2|x_1 x_2 \lambda)$ can be signalling!

Free operations: wirings and stochastic 1-way LOCC

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free operation on free resource gives non-free assemblage

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Need for a principled approach to 'freeness'

Conclusions and outlook

- Traditional and generalised EPR scenarios
- Causal perspective on their 'object of interest' (assemblages)
- Causally-underpinned resource theory:
  free = classical common cause (LOSR)
- (i) Traditional scenarios + quantum enveloping theory
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- Type-changing free operations: how do pre-orders relate?
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Diversity is a fact
Inclusion is an act

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