Quantum Suplattices

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A very short summary

In the context of supremum-preserving functions as morphisms, complete lattices are called suplattices.

We introduce a noncommutative version of complete lattices, which we call quantum suplattices, which:

- are obtained via a scheme called discrete quantization;
- are algebras for monads that are quantum versions of the power set monad and the lower set monad;
- are not generalizations of ordinary suplattices;
- satisfy usual theorems for ordinary suplattices such as the existence of Galois connections and the Knaster-Tarski Theorem.

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Non-commutative mathematics

- A program to obtain natural models of quantum structures;
- Main idea: algebras of operators on a Hilbert space H can be used to construct 'non-commutative' generalizations of classical structures;
- Example: $X \mapsto C(X)$ yields a categorical duality between the categories of compact Hausdorff spaces and of commutative unital C*-algebras (Gelfand duality);
- Hence the dual of the category of unital C*-algebras can be regarded as the category of 'non-commutative' compact Hausdorff spaces.
- Quantization is the process of finding noncommutative versions of a mathematical structure.

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Background

- Duan, Severini, Winter: quantum graphs in quantum error correction;
- Kuperberg and Weaver: quantization of metric spaces; quantum hamming metric in quantum error correction
- Weaver: identification of quantum relations as underlying structure of quantum metric spaces and quantum graphs;
- Weaver: quantum posets;
- Kornell: quantum sets and their categorical properties;
- Kornell, L., Mislove: categorical structure of quantum posets;
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Motivation for the quantization of suplattices

- Topologies on sets are examples of suplattices
- There is no theory of noncommutative topological spaces beyond locally compact Hausdorff spaces;
- Several relevant topologies such as the Scott topology on a cpo are not locally compact or Hausdorff;
- Quantum suplattices might be a first step towards a theory of quantum topological spaces beyond locally compact Hausdorff spaces;
- Suplattices form an example of a $*$-autonomous category; such categories can be used for the semantics of classical multiplicative linear logic;
- We expect that also quantum suplattices form a $*$-autonomous category.

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Internalization

Internalization is the process of generalizing set-theoretic constructions that can be defined in terms of the categorical structure of **Set** or **Rel** to other categories that posses the same categorical structure needed for these constructions. Example: in any category with all finite products, a group G is an object equipped with morphisms $m : G \times G \to G$, $e : 1 \to G$, and $(-)^{-1} : G \to G$ such that:

- Unitality:

$$\begin{array}{ccc} G \times 1 & \xrightarrow{\mathrm{id}_G \times e} & G \times G \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle m} \\ G & \xrightarrow[=]{} & G \end{array} \qquad \begin{array}{ccc} 1 \times G & \xrightarrow{e \times \mathrm{id}_G} & G \times G \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle m} \\ G & \xrightarrow[=]{} & G \end{array}$$

- Associativity:

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- Inverses:

$$\begin{array}{ccccc} G & \xrightarrow{\mathrm{diag}_G} & G \times G & \xrightarrow{\mathrm{id}_G \times (-)^{-1}} & G \times G \\ {\scriptstyle !}\downarrow & & & & \downarrow{\scriptstyle m} \\ 1 & & \xrightarrow[e]{} & & G \end{array} \qquad \begin{array}{ccccc} G & \xrightarrow{\mathrm{diag}_G} & G \times G & \xrightarrow{(-)^{-1} \times \mathrm{id}_G} & G \times G \\ {\scriptstyle !}\downarrow & & & & \downarrow{\scriptstyle m} \\ 1 & & \xrightarrow[e]{} & & G \end{array}$$

Groups in **Top** are topological groups, groups in **SmoothManifolds** are Lie groups.

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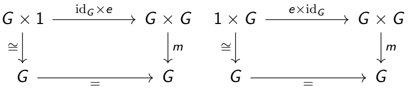

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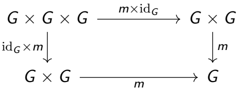

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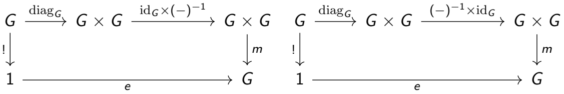

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Quantization by internalization

- We employ a method of quantization by internalizing structures in a suitable category of C*-algebras whose objects are noncommutative generalizations of sets;
- In general, one can internalize functions in a category resembling **Rel**, whereas binary relations cannot always be internalized in a category resembling **Set**;
- Therefore, our category of operator algebras should be a noncommutative generalization of the category **Rel**;
- The dual of the category **WStar** of von Neumann algebras can be regarded the category of 'non-commutative' measure spaces.
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Hereditarily atomic von Neumann algebras

- Hereditarily atomic von Neumann algebras are von Neumann algebras isomorphic to $\bigoplus_{i \in I} L(H_i)$ with H_i a finite-dimensional Hilbert space, and can be used as non-commutative generalizations of sets;
- The category **WRel** of von Neumann algebras and quantum relations is a quantaloid (**Sup**-enriched category) with a dagger;
- Its full subcategory $\mathbf{WRel}_{\mathrm{HA}}$ of hereditarily atomic von Neumann algebras is a dagger compact quantaloid just like **Rel**.
- Discrete quantization is the process of internalizing mathematical structures in $\mathbf{WRel}_{\mathrm{HA}}$;
- Compare: fuzzification can be regarded as internalizing structures in V-**Rel** for a quantale V such as $[0, 1]$;
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Quantum sets and binary relations

Definition

- A quantum set $\mathcal{X}$ is a family of finite-dimensional Hilbert spaces called atoms of $\mathcal{X}$;
- A binary relation $R : \mathcal{X} \to \mathcal{Y}$ is a function assigning to each atom X of $\mathcal{X}$ and each atom Y of $\mathcal{Y}$ a subspace $R(X, Y)$ of the space $L(X, Y)$ of linear maps $X \to Y$.

A binary relation R from $\mathcal{X} = \{X_1, \ldots, X_n\}$ to $\mathcal{Y} = \{Y_1, \ldots, Y_m\}$ can be represented in matrix form:

$$R = \begin{bmatrix} R(X_1, Y_1) & R(X_2, Y_1) & \cdots & R(X_n, Y_1) \\ R(X_1, Y_2) & R(X_2, Y_2) & \cdots & R(X_n, Y_2) \\ \vdots & \vdots & \ddots & \vdots \\ R(X_1, Y_m) & R(X_2, Y_n) & \cdots & R(X_n, Y_m) \end{bmatrix}$$

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Quantum sets and binary relations

- Any quantum set $\mathcal{X}$ corresponds to a hereditarily atomic von Neumann algebra $\bigoplus_{X\in\mathcal{X}} L(X)$ that is unique up to isomorphism;
- The identity relation $I_{\mathcal{X}}$ on $\mathcal{X}$ is the 'diagonal' matrix with diagonal elements $I_{\mathcal{X}}(X,X)=\mathbb{C}1_X$;
- Binary relations on quantum sets were introduced by Kornell[1], but are essentially Weaver's quantum relations on von Neumann algebras[2];
- The category **qRel** of quantum sets and binary relations is dagger-compact;
- The inclusion relation between subspaces induces an order $\leq$ on binary relations between $\mathcal{X}$ and $\mathcal{Y}$ such that **qRel** becomes a quantaloid;
- We have a fully faithful functor $`(-)$: **Rel** $\to$ **qRel** preserving the dagger structure and the order between relations.

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Quantum sets and binary relations

- Any quantum set $\mathcal{X}$ corresponds to a hereditarily atomic von Neumann algebra $\bigoplus_{X \in \mathcal{X}} L(X)$ that is unique up to isomorphism;
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A morphism $f : X \to Y$ in **Rel** is a function if and only if $f^\dagger \circ f \geq 1_X$ and $f \circ f^\dagger \leq 1_Y$.

Definition

A function $F : \mathcal{X} \to \mathcal{Y}$ between quantum sets is a relation satisfying $F^\dagger \circ F \geq I_\mathcal{X}$ and $F \circ F^\dagger \leq I_\mathcal{Y}$. The category of quantum sets and functions is denoted by **qSet**.

- **qSet** is complete, cocomplete and symmetric monoidal closed[3];
- The assignment $\mathcal{X} \mapsto \ell^\infty(\mathcal{X}) := \bigoplus_{X \in \mathcal{X}} L(X)$ extends to a duality between **qSet** and the category $\mathbf{WStar}_{\mathrm{HA}}$ of hereditarily atomic von Neumann algebras and normal unital $*$-homomorphisms;
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Quantum posets

Definition

- An preorder on a quantum set $\mathcal{X}$ is a binary relation $\preccurlyeq : \mathcal{X} \to \mathcal{X}$ such that
  (1) $I_{\mathcal{X}} \leq \preccurlyeq$ (reflexivity);
  (2) $\preccurlyeq \circ \preccurlyeq \leq \preccurlyeq$ (transitivity).
- The opposite $\succcurlyeq := \preccurlyeq^{\dagger}$ of a preorder is a preorder.
- A preorder $\preccurlyeq$ on $\mathcal{X}$ is called an order if
  (3) $\preccurlyeq \wedge \succcurlyeq \leq I_{\mathcal{X}}$ (antisymmetry)
- A function $F : (\mathcal{X}, \preccurlyeq_{\mathcal{X}}) \to (\mathcal{Y}, \preccurlyeq_{\mathcal{Y}})$ is monotone if $F \circ \preccurlyeq_{\mathcal{X}} \leq \preccurlyeq_{\mathcal{Y}} \circ F$

Example

Let $\mathcal{H}_2$ be the quantum set whose single atom is the two-dimensional Hilbert space H_2. Then $(\mathcal{H}_2, \preccurlyeq)$ is a quantum poset for

$$\preccurlyeq(H_2, H_2) := \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

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Lower sets, suplattices and monotone relations

(1) Suplattices are the algebras of the lower set monad D on **Pos**;

(2) Suplattices are posets X such that the canonical embedding $X \to D(X)$, $x \mapsto \downarrow x$ has a lower Galois adjoint $\bigvee$.

Definition

A monotone relation $r : X \to Y$ between posets is a binary relation such that $(x_1, y_1) \in r$ implies $(x_2, y_2) \in r$ for each $x_1 \leq x_2$ in X and each $y_1 \geq y_2$ in Y.

- Any monotone relation $r : X \to Y$ corresponds to a monotone function $X^{\mathrm{op}} \times Y \to 2$, so to a 2-enriched profunctor when X and Y are regarded as 2-enriched categories;
- The category **RelPos** of posets and monotone relations is compact closed.
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Monotone relations between quantum posets

Definition

A monotone relation $R : (\mathcal{X}, \preccurlyeq_{\mathcal{X}}) \to (\mathcal{Y}, \preccurlyeq_{\mathcal{Y}})$ between quantum posets is a binary relation $R : \mathcal{X} \to \mathcal{Y}$ such that $\succcurlyeq_{\mathcal{Y}} \circ R \leq R$ and $R \circ \succcurlyeq_{\mathcal{X}} \leq R$.

Theorem

The category **qRelPos** *of quantum posets and monotone relations is compact closed.*

Theorem

The embedding **qPos** $\to$ **qRelPos** *has a right adjoint; its induced monad* $\mathcal{D}$ *is called the quantum lower set monad.*

- The existence of right adjoints of embeddings **Pos** $\to$ **RelPos**, **Rel** $\to$ **Set**, **qRel** $\to$ **qSet** and **qPos** $\to$ **qRelPos** can all be proven in one scheme involving the embedding of a symmetric monoidal closed category **S** into a compact closed category **R**;
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Galois connections between quantum posets

Definition

The pointwise order $\sqsubseteq_{\mathcal{Y}}$ of functions $F, G : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}$ is a quantum set and $\mathcal{Y}$ is a quantum poset ordered by $\preccurlyeq$ is defined by $F \sqsubseteq_{\mathcal{Y}} G$ if and only if $F \leq \succcurlyeq \circ G$.

Definition

A Galois connection between quantum posets $(\mathcal{X}, \preccurlyeq_{\mathcal{X}})$ and $(\mathcal{Y}, \preccurlyeq_{\mathcal{Y}})$ consists of a pair of monotone maps $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{Y} \to \mathcal{X}$ such that

$$I_{\mathcal{X}} \sqsubseteq_{\mathcal{X}} G \circ F, \qquad F \circ G \sqsubseteq_{\mathcal{Y}} I_{\mathcal{Y}}.$$

F is called the lower Galois adjoint of G.

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Quantum suplattices

The unit of the **qPos**/**qRelPos** adjunction yields a canonical order embedding $\mathcal{X} \to \mathcal{D}(\mathcal{X})$; a quantum generalization of the order embedding $X \to D(X)$, $x \mapsto \downarrow x$ for ordinary posets X.

Definition

A quantum poset $(\mathcal{X}, \preccurlyeq_{\mathcal{X}})$ is called a quantum suplattice if the canonical order embedding $\mathcal{X} \to \mathcal{D}(\mathcal{X})$ has a lower Galois adjoint $\bigvee_{\mathcal{X}}$. A monotone map $F : (\mathcal{X}, \preccurlyeq_{\mathcal{X}}) \to (\mathcal{Y}, \preccurlyeq_{\mathcal{Y}})$ between quantum suplattices is called a sup-homomorphism if $F \circ \bigvee_{\mathcal{X}} = \bigvee_{\mathcal{Y}} \circ \mathcal{D}(F)$. The category of quantum suplattices and sup-homomorphisms is denoted by **qSup**.

Example

Let $\mathcal{X}$ be a quantum poset. Then $\mathcal{D}(\mathcal{X})$ is a quantum suplattice where $\bigvee_{\mathcal{D}(\mathcal{X})}$ is the multiplication $\mathcal{D}^2(\mathcal{X}) \to \mathcal{D}(\mathcal{X})$.

Theorem

qSup *is equivalent to the Eilenberg-Moore category of* $\mathcal{D}$.

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A quantum poset $(\mathcal{X}, \preccurlyeq_{\mathcal{X}})$ is called a quantum suplattice if the canonical order embedding $\mathcal{X} \to \mathcal{D}(\mathcal{X})$ has a lower Galois adjoint $\bigvee_{\mathcal{X}}$. A monotone map $F : (\mathcal{X}, \preccurlyeq_{\mathcal{X}}) \to (\mathcal{Y}, \preccurlyeq_{\mathcal{Y}})$ between quantum suplattices is called a sup-homomorphism if $F \circ \bigvee_{\mathcal{X}} = \bigvee_{\mathcal{Y}} \circ \mathcal{D}(F)$. The category of quantum suplattices and sup-homomorphisms is denoted by **qSup**.

Example

Let $\mathcal{X}$ be a quantum poset. Then $\mathcal{D}(\mathcal{X})$ is a quantum suplattice where $\bigvee_{\mathcal{D}(\mathcal{X})}$ is the multiplication $\mathcal{D}^2(\mathcal{X}) \to \mathcal{D}(\mathcal{X})$.

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qSup *is equivalent to the Eilenberg-Moore category of* $\mathcal{D}$.

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Suplattices are not quantum suplattices

Proposition

The fully faithful functor $`(-) : \mathbf{Pos} \to \mathbf{qPos}$ *does not restrict and corestrict to a functor* $\mathbf{Sup} \to \mathbf{qSup}$.

Counterexample

The 4-element Boolean algebra is not a quantum suplattice.

- If X is a poset with poset $D(X)$ of lower sets, then $`X$ is a quantum poset, and $`D(X)$ is a quantum poset which embeds into $\mathcal{D}(`X)$;
- The image of this embedding are the one-dimensional atoms of $\mathcal{D}(`X)$, i.e., its classical part of $\mathcal{D}(`X)$.
- However, $\mathcal{D}(`X)$ has also higher-dimensional atoms.

Conjecture

Let $(X, \sqsubseteq)$ *be a complete linearly ordered lattice. Then* $(`X, `\sqsubseteq)$ *is a weak quantum suplattice, hence a quantum suplattice.*

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Quantum versions of some theorems on suplattices

Theorem

The opposite $(\mathcal{X}, \succcurlyeq)$ of a quantum suplattice $(\mathcal{X}, \preccurlyeq)$ is a quantum suplattice.

Theorem

Let $F : \mathcal{X} \to \mathcal{Y}$ be a monotone map between quantum suplattices. Then F is a sup-homomorphism if and only if F is a lower Galois adjoint.

Let $F : \mathcal{X} \to \mathcal{X}$ be a monotone endomap on a quantum poset $\mathcal{X}$. A subset $\mathcal{Y} \subseteq \mathcal{X}$ with canonical embedding $J_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{X}$ is called a subset of fixpoints if $F \circ J_{\mathcal{Y}} = J_{\mathcal{Y}}$.

Theorem (Quantum Knaster-Tarski)

Let $F : \mathcal{X} \to \mathcal{X}$ be a monotone endomap on a quantum suplattice $(\mathcal{X}, \preccurlyeq)$. Then the largest subset of fixpoints $\mathcal{Y}$ of $\mathcal{X}$ exists and is a quantum suplattice in its relative order $J_{\mathcal{Y}}^{\dagger} \circ \preccurlyeq \circ J_{\mathcal{Y}}$.

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Future work

Conjecture (Quantum Cantor–Schröder–Bernstein)

Let $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{Y} \to \mathcal{X}$ be injective functions between quantum sets $\mathcal{X}$ and $\mathcal{Y}$. Then there is a bijection $\mathcal{X} \cong \mathcal{Y}$.

In terms of operator algebras, this translates to

Conjecture

Let $f : M \to N$ and $g : N \to M$ be surjective normal unital $*$-homomorphisms between hereditarily atomic von Neumann algebras M and N. Then there is a $*$-isomorphism $M \to N$.

Probably we need:

Conjecture

Let $\mathcal{X}$ and $\mathcal{Y}$ be quantum posets for which there is an order isomorphism $\mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y})$. Then there is an order isomorphism $\mathcal{X} \to \mathcal{Y}$.

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