

Every non-signalling channel is common-cause realizable

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In this work, we investigate the relation between the two options.

Outline

- 1 Preliminaries
 - Generalized probabilistic theories (GPTs)
 - Non-signalling channels
 - Common-cause realizations
- 2 Setting up the problem
 - Common-cause completions
- 3 Our construction
 - Overview
- 4 Final remarks

Generalized probabilistic theories

(focus on compositionality)

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*Causal, locally-tomographic

Compositional Structure

Symmetric monoidal category: **diagrams**

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Systems:



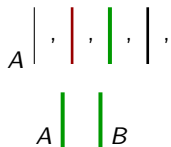
Processes:

States:

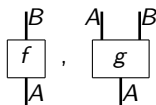
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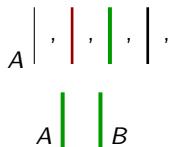


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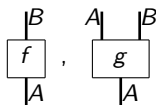
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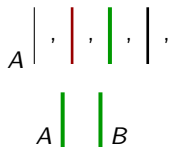
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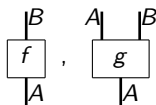
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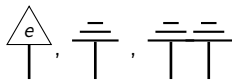
Processes:



States:



Effects:

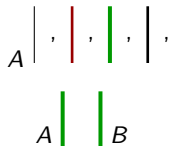


Convex structure:

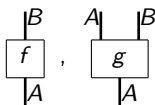
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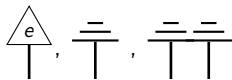
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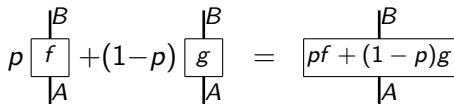
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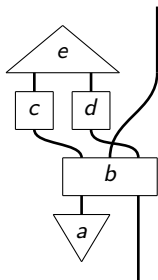


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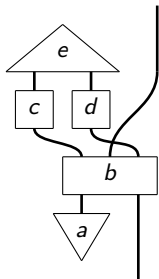
Complex Diagrams:



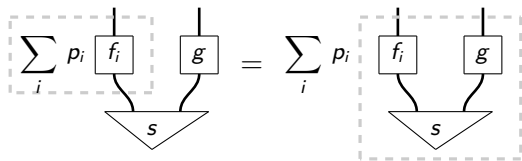
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Example: **Stoch**

Real vector spaces, stochastic maps, tensor and matrix products:

$$| = \mathbb{R}^2$$

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$$\begin{array}{c} \begin{array}{|} \hline B \\ \hline \end{array} \\ \hline \begin{array}{|} \hline f \\ \hline \end{array} \\ \hline \begin{array}{|} \hline A \\ \hline \end{array} \end{array} = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix}$$

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$$\begin{array}{c} | \\ \nabla \\ s' \end{array} \begin{array}{c} | \\ \nabla \\ s \end{array} = \begin{pmatrix} 1/2 \\ 2/3 \end{pmatrix} \otimes \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

Definition of Causal, Locally Tomographic GPT

With the composition rules in mind:

- 1 The SMC contains **Stoch** as a full subtheory.

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Let's discuss the last two points

Equality of processes

Tomography

$$\begin{array}{c} |B \\ \boxed{f} \\ |A \end{array} = \begin{array}{c} |B \\ \boxed{g} \\ |A \end{array} \iff \forall T, X, Y, \begin{array}{c} |Y \\ \boxed{\begin{array}{c} |B \\ \boxed{f} \\ |A \end{array}} \\ \tau \\ \boxed{\phantom{\begin{array}{c} |B \\ \boxed{f} \\ |A \end{array}}} \\ |X \end{array} = \begin{array}{c} |Y \\ \boxed{\begin{array}{c} |B \\ \boxed{g} \\ |A \end{array}} \\ \tau \\ \boxed{\phantom{\begin{array}{c} |B \\ \boxed{g} \\ |A \end{array}}} \\ |X \end{array} \quad (1)$$

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Local Tomography

$$\begin{array}{c} |B \\ \square \\ f \\ |A \end{array} = \begin{array}{c} |B \\ \square \\ g \\ |A \end{array} \iff \forall s, M, Y, \begin{array}{c} |Y \\ \square \\ M \\ |B \\ \square \\ f \\ |A \\ \square \\ \triangle \\ s \end{array} = \begin{array}{c} |Y \\ \square \\ M \\ |B \\ \square \\ g \\ |A \\ \square \\ \triangle \\ s \end{array} \quad (2)$$

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*Equality by operational equivalence

Unique discarding effect

$$\overline{\overline{A|B}} = \overline{\overline{A}} \overline{\overline{B}}, \quad (3)$$

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Unique effect

$$\overline{\overline{\triangle e|}} = \overline{\overline{A|}} \quad (4)$$

Unique discarding effect

$$\overline{\overline{A|B}} = \overline{\overline{A}} \overline{\overline{B}}$$

(3)

Unique effect

$$\overline{\overline{e}} = \overline{\overline{A}}$$

(4)

Our probabilities come from inside the stochastic maps and state vectors

With the context of GPTs in mind, we can now define the notions of **common-cause realisation** and **non-signalling**.

Non-signalling channels

Intuition

We want to encode the impossibility of signalling diagrammatically.

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Non-signalling from AC to BD:

$$\begin{array}{c} \text{=} \\ \text{=} \\ \text{c} \\ \hline \Lambda \\ \text{A} \quad \text{B} \end{array} = \begin{array}{c} \text{=} \\ \text{=} \\ \text{=} \\ \hline \Lambda_b \\ \text{A} \quad \text{B} \end{array} \quad (5)$$

The diagram shows an equality between two circuit-like representations. On the left, a box labeled Λ has two input wires from below labeled A and B . Above the box, there is a wire labeled c that is connected to the top of the box. Above the c wire, there are two horizontal lines, with the top one being a double line. To the right of the box, a wire labeled D extends upwards. On the right side of the equation, a box labeled Λ_b has two input wires from below labeled A and B . Above the box, there are three horizontal lines, with the top one being a double line. To the right of the box, a wire labeled D extends upwards.

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since

$$\begin{array}{c} \text{---} \\ \text{c} \\ \text{---} \\ \Lambda \\ \text{---} \\ A \\ \text{---} \\ f \\ \text{---} \\ A \end{array} \quad \begin{array}{c} | \\ D \end{array} = \begin{array}{c} \text{---} \\ A \\ \text{---} \\ \Lambda_b \\ \text{---} \\ A \\ \text{---} \\ f \\ \text{---} \\ A \end{array} \quad \begin{array}{c} | \\ D \end{array} = \begin{array}{c} \text{---} \\ A \\ \text{---} \\ \Lambda_b \\ \text{---} \\ A \quad B \end{array} \quad \begin{array}{c} | \\ D \end{array} \quad (6)$$

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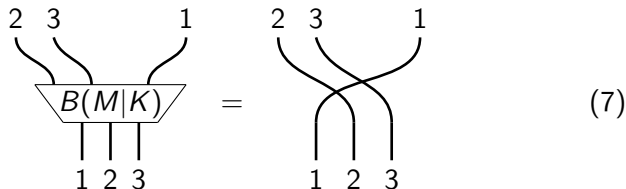
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A channel is NS if it can't signal between any two parties.

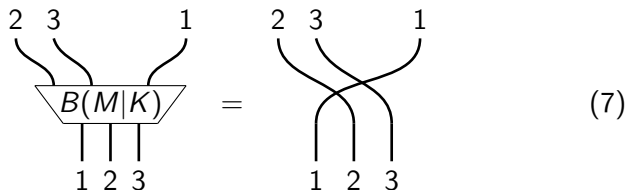
Definition: Multipartite case

Consider $B(M|K)$ are bipartitions of wires such as



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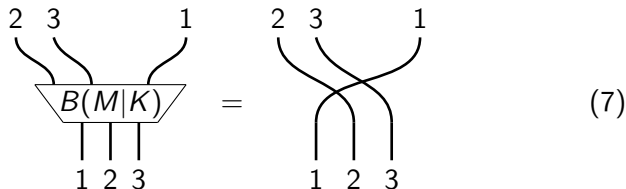
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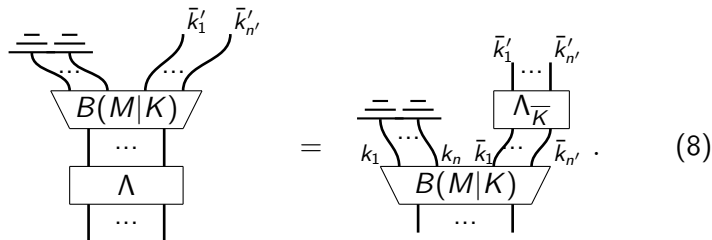
Then, Λ is non-signalling iff **for all such bipartitions**

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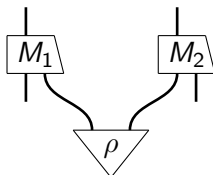
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Common-cause realization

Prototypical example

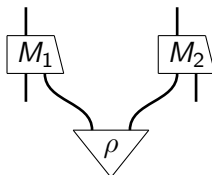
Bell scenario



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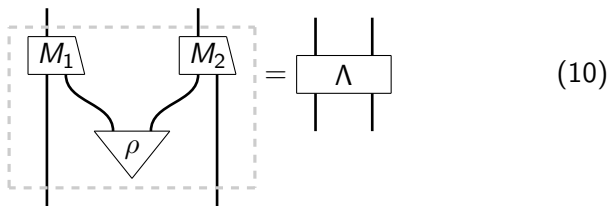


(9)

The state s can be seen as a **common-cause** for the two measurements

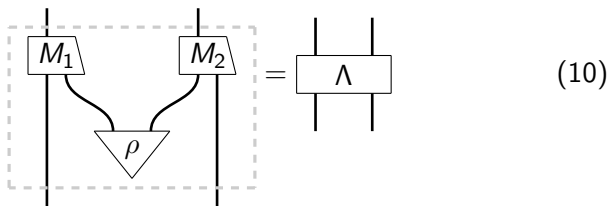
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Then ρ, M_1, M_2 provide a **common-cause decomposition** of Λ .

Common-cause decomposition

$$\begin{array}{c}
 1' \quad \dots \quad |N' \\
 \boxed{\Lambda} \\
 1 \quad \dots \quad |N
 \end{array}
 =
 \begin{array}{c}
 1' \quad \dots \quad |N' \\
 \boxed{T_1} \quad \dots \quad \boxed{T_n} \\
 \begin{array}{c}
 | \\
 1'' \\
 \dots \\
 |N'' \\
 \dots \\
 |N
 \end{array} \\
 \begin{array}{c}
 \triangle S \\
 \dots
 \end{array}
 \end{array}
 \quad (11)$$

Is a common-cause decomposition of Λ .

Common-cause decomposition

The diagram illustrates the common-cause decomposition of a channel Λ . On the left, a box labeled Λ has input wires labeled $1'$, \dots , N' and output wires labeled 1 , \dots , N . This is set equal to a more complex circuit on the right. The circuit starts with a wire labeled $1'$ that goes straight down to a wire labeled 1 . A second wire, labeled $1''$, branches off from the $1'$ wire and goes into a trapezoidal box labeled T_1 . This wire then continues down to the output wire 1 . A similar structure is shown for the other inputs: a wire labeled N' goes into a trapezoidal box labeled T_n , and another wire labeled N'' branches off from the N' wire and goes into a triangular box labeled S . The output of S is labeled N . Ellipses between T_1 and T_n indicate intermediate boxes. The entire decomposition is labeled with the equation number (11) on the right.

$$\Lambda = \begin{array}{c} 1' \dots N' \\ \boxed{\Lambda} \\ 1 \dots N \end{array} = \begin{array}{c} 1' \dots N' \\ \begin{array}{c} \boxed{T_1} \dots \boxed{T_n} \\ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \\ \begin{array}{c} 1'' \\ \dots \\ N'' \end{array} \\ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \\ \begin{array}{c} 1 \\ \dots \\ N \end{array} \end{array} \end{array} \quad (11)$$

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Note that this implies Λ is non-signalling.

If we don't have a common-cause decomposition, we can still ask whether it exists within **some** other theory.

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GPT-common-cause realisable channel

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GPT-common-cause realisable channel

$\Lambda \in \mathbf{G}$ is GPT-common-cause realisable iff there exists a theory such that

The diagram shows an equality between two expressions. On the left is a box labeled Λ with input wires labeled $1', \dots, N'$ and output wires labeled $1, \dots, N$. On the right is a sequence of boxes labeled T_1, \dots, T_n connected in series. The input wires $1', \dots, N'$ enter from the top. The output wires $1, \dots, N$ exit from the bottom. Red lines represent causal connections between the boxes, showing the flow of information from the theories T_i to a state S (represented by a downward-pointing triangle) which then influences the outputs. The label (12) is to the right of the diagram.

$$\Lambda \quad = \quad T_1 \dots T_n \quad S \quad (12)$$

and the original theory \mathbf{G} is a **full subtheory** of the new one.

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What to keep in mind:

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- 2 We work with a definition analogous to quantum theory with only CPTP maps.
- 3 GPTs have a diagrammatic calculus.
- 4 We diagrammatically define non-signalling with the discarding effect.
- 5 We diagrammatically define common-cause realisations with shared states.

The problem

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- But it is common-cause realisable with quantum theory.
- A PR-box correlation is not common-cause realisable in quantum theory.
- But it is realisable in boxworld.

So we can say **classical** and **quantum** are **not common-cause complete**. Moreover, **boxworld** common-cause decomposes **all** non-signalling classical channels.

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- A bell correlation is not common-cause realisable in classical theory.
- But it is common-cause realisable with quantum theory.
- A PR-box correlation is not common-cause realisable in quantum theory.
- But it is realisable in boxworld.

So we can say **classical** and **quantum** are **not common-cause complete**. Moreover, **boxworld** common-cause decomposes **all** non-signalling classical channels.

Boxworld is a **common-cause completion** of classical theory.

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So, we **generalise that observation** to formulate our problem.

The problem

We want to know whether:

- Given a **causal, locally tomographic GPT \mathbf{G}** ,

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- Such that for all **non-signalling** channels Λ in \mathbf{G} ,
- Λ are **common-cause realisable** in \mathbf{G}' .

In one sentence:

Can we always find a common-cause completion of a causal locally tomographic GPT?

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Yes

Common-cause completions

We answer the question by providing a construction \mathcal{C} that takes a GPT \mathbf{G} and outputs a GPT $\mathcal{C}[\mathbf{G}]$ that is a common-cause completion of \mathbf{G} .

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Let's look at the basic idea of the construction.

Common-cause completions

A previous result¹, guarantees we can always write

$$\begin{array}{c} 1'^{\wedge} \dots m'^{\wedge} \\ \boxed{\Lambda} \\ 1^{\wedge} \dots m^{\wedge} \end{array} = \begin{array}{c} \dots \\ \tilde{\eta}_1^{\wedge} \quad \tilde{\eta}_m^{\wedge} \\ \dots \\ \tilde{i}_1^{\wedge} \dots \tilde{i}_m^{\wedge} \\ \blacktriangledown \\ \tilde{\xi}^{\wedge} \end{array} \quad (13)$$

Where $\tilde{\xi}^{\wedge}$ is an (unphysical) affine combination of states from the GPT.

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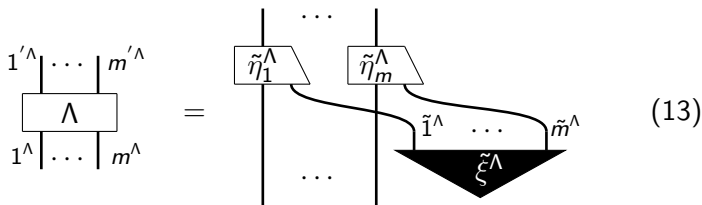
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Why? Negative probabilities

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If we can find a way to add processes $\tilde{\xi}^\Lambda, \tilde{\eta}_i^\Lambda$ to the GPT in a consistent way, then we can use that theorem to construct a common-cause completion.

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So, **let's instead add**

$$\eta_i^\Lambda \quad := \quad \tilde{\eta}_i^\Lambda \quad (14)$$

and

$$\xi^\Lambda \quad := \quad \tilde{\xi}^\Lambda \quad (15)$$

Now, we can rewrite the theorem decomposition as

$$\begin{array}{c}
 1'^{\wedge} \quad \dots \quad m'^{\wedge} \\
 | \quad \quad \quad | \\
 \boxed{\Lambda} \\
 | \quad \quad \quad | \\
 1^{\wedge} \quad \dots \quad m^{\wedge}
 \end{array}
 =
 \begin{array}{c}
 1'^{\wedge} \quad \dots \quad m'^{\wedge} \\
 | \quad \quad \quad | \\
 \boxed{\eta_1^{\wedge}} \quad \quad \quad \boxed{\eta_m^{\wedge}} \\
 | \quad \quad \quad | \\
 1^{\wedge} \quad \dots \quad m^{\wedge}
 \end{array}
 \begin{array}{c}
 \quad \quad \quad A_1^{\wedge} \quad \dots \quad A_m^{\wedge} \\
 \quad \quad \quad \searrow \quad \quad \quad \swarrow \\
 \quad \quad \quad \xi^{\wedge}
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 \text{---} \quad \text{---} \quad \text{---} \\
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And we can't freely compose ξ^{\wedge} with other processes without going through η_i^{\wedge} first

This is the key idea of the construction.

From this starting point, the challenge is to **guarantee all the compositional properties** after the extension from \mathbf{G} to $\mathbf{G} \sqcup \eta$.

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Then, $\mathcal{C}[\mathbf{G}] = \text{Conv}[\overline{\mathbf{G} \sqcup \eta}] / \sim$ should be a common-cause completion of \mathbf{G} .

Each of those steps requires a proof that it **guarantees the desired properties, and preserves the ones from previous steps**

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We prove that indeed $\mathcal{C}[\mathbf{G}] = \text{Conv}[\overline{\mathbf{G} \sqcup \eta}] / \sim$ is a causal GPT and is a common-cause completion of \mathbf{G} .

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That leads us to our main result

Main result

Theorem V.1. Given a locally-tomographic causal GPT \mathbf{G} , its set of multipartite non-signalling channels is the same as its set of multipartite common-cause realisable channels.

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Note these common-causes might not be state-preparations allowed in \mathbf{G} .

Outlook

- We investigate the relation between the **non-signalling** channels and the **GPT-common-cause realisable** channels of **causal, locally tomographic GPTs**.

²D. Schmid, H. Du, M. Mudassar, G. Coulter-de Wit, D. Rosset, and M. J. Hoban, Quantum 5, 419 (2021)

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Outlook

- We investigate the relation between the **non-signalling** channels and the **GPT-common-cause realisable** channels of **causal, locally tomographic GPTs**.
- We show that, in fact, **the two sets coincide**.

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 - $NS = GPT\text{-}CCC$
 - There exists a GPT that realizes all non-signalling assemblages
- Our result provides a more principled reason to use the non-signalling channels as the enveloping theory in resource theories.

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Open Questions

- 1 We know $\mathcal{C}[\mathbf{G}]$ is causal, but is it locally tomographic?

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- 2 Can we find a similar scheme for when \mathbf{G} is not locally tomographic?

Thank you!

Take home message:

- **Non-signalling** channels coincide with **GPT-common-cause realisable** channels in **causal, locally tomographic GPTs**.
- This provides a causal justification for using non-signalling channels as the enveloping theory in resource theories.

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