Graph approach to generate multipartite entanglement in linear quantum systems

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18th of July, 2023

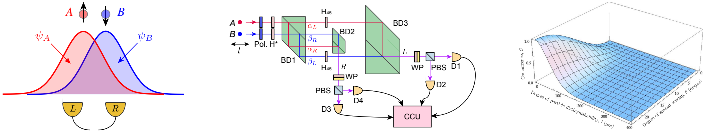
References

- Chin, Seungbeom, Yong-Su Kim, and Sangmin Lee. "Graph picture of linear quantum networks and entanglement." Quantum 5 (2021): 611.
- Chin, Seungbeom, Yong-Su Kim, and Marcin Karczewski. "Shortcut to Multipartite Entanglement Generation: A Graph Approach to Boson Subtractions", arXiv preprint arXiv:2211.04042 (2022).
- Chin, Seungbeom, "From linear quantum system graphs to qubit graphs: Heralded generation of graph states", arXiv preprint, arXiv:2306.15148 (2023).

Overview

Entanglement generation with identical particles in linear quantum systems (LQSs)

- Particle indistinguishability + Spatial overlap → Entanglement



(Barros, SC, Pramanik, Lim, Cho, Huh, & Kim, Optics Express, 2020)

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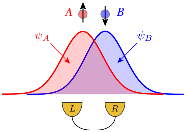

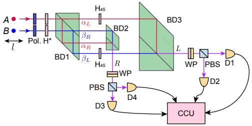

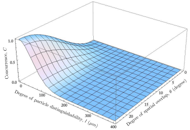

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Probabilistic entanglement generations w/ bosons

Postselection

- Prearrangement on which state to "postselect"
- Requires detection of all states in the circuit, which will contain unwanted states
- In general, cannot be directly used as quantum gates

Heralding

- Employs ancillary single bosons and modes as "heralds" of the expected target states
- Allows for sorting out the experimental runs for the target states w/o measuring them
- needs more particles and modes, complicated to design

Bipartite case is OK, but...

$$\text{QUESTION: LQSs} \overset{?}{\Longleftrightarrow} N\text{-partite entangled states}$$

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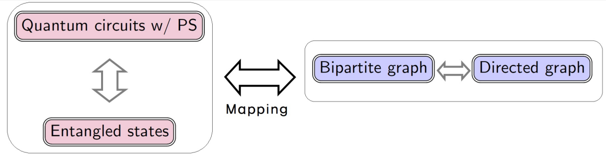
- What we have done so far: found necessary conditions for LQSs to generate genuine entanglement and actually found several simple schemes
- Our ultimate goal: Give a straightforward path to construct optimal LQSs that generate specific genuinely entangled states

Postselected schemes: SC, Y.S. Kim, & S. Lee, Quantum 5 (2021), 611



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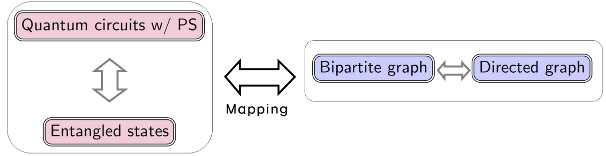
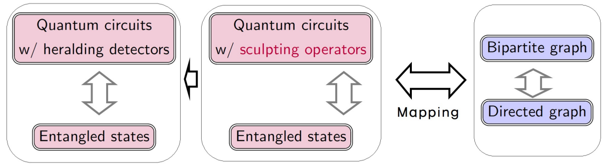


Heralded schemes: SC, Y.S. Kim, & M. Karczewski, arXiv:2211.04042



Sculpting protocol

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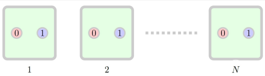
1. Initial state: We prepare the maximally symmetric state $|Sym_N\rangle$ of $2N$ bosons, i.e., each boson has different states (either spatial or internal) with each other.



$$|Sym_N\rangle \equiv \prod_{j=1}^{N} (\hat{a}^{\dagger}_{j,0}\hat{a}^{\dagger}_{j,1})|vac\rangle$$

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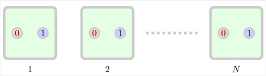

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2. Operation: We apply the sculpting operator $\hat{A}_N$

$$\hat{A}_N \equiv \prod_{l=1}^{N} \hat{A}^{(l)} \equiv \prod_{l=1}^{N} \left(\sum_{j=1}^{N} \alpha_j^{(l)} \hat{a}_{j,\psi_j^{(l)}} \right)$$

to the initial state $|Sym_N\rangle$. The sculpting operator must be set to extract one boson per mode (no-bunching restriction).

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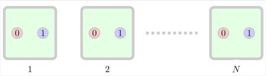

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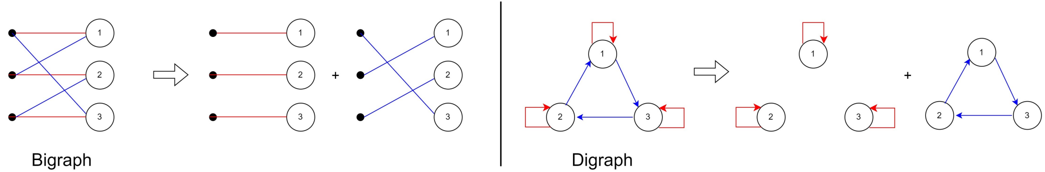
3. Final state: The final state $|\Psi\rangle_{fin} = \hat{A}_N|Sym_N\rangle$ can be fully separable, partially separable, or genuinely entangled.

A very short glossary in graph theory

- Graph $G = (V, E)$: A collection of a vertex set V and an edge set E. Each edge can have a color and a weight.
- Bipartite graph (bigraph) $G_b = (U \cup V, E)$: Two disjoint vertex sets U & V. Edges connect U and V. G_b is balanced if $|U| = |V|$ (G_{bb})
- Perfect matching (PM) in G_{bb}: One-to-one connection between U and V
- A balanced bigraph G_{bb} can always be drawn as a directed graph (digraph) G_d



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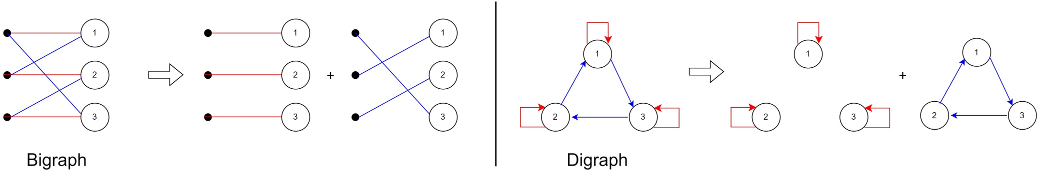


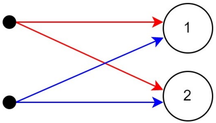
Fundamental elements of bosonic systems

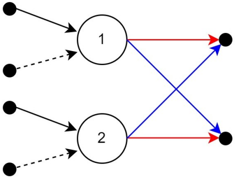
- Boson creation/annihilation operators
- Spatial modes (subsystems)
- Dynamical relations b/w particles and modes

Bosonic systems → Graphs

Linear quantum system (LQS)	Directed bipartite Graph $G_t = (U \cup V, E)$
Spatial modes	Labelled vertices $ⓘ \in U$
Creation operators	Unlabelled vertices $(\bullet \in V)$ w/ incoming edges
Annihilation operators	Unlabelled vertices $(\bullet \in V)$ w/ outgoing edges
Spatial distributions of operators	Directed edges $\in E$
Probability amplitude $\alpha_j^{(l)}$	Edge weight $\alpha_j^{(l)}$
Internal state $\psi_j^{(l)}$	Edge weight $\psi_j^{(l)}$ (sometimes replaced with colors)

(N=2 example)





Postselected schemes (w/o annihilation operators)

SC, Y.S. Kim, & S. Lee, Quantum 5 (2021): 611.

Linear quantum system (LQS)	**Bipartite Graph** $G_b = (U \cup V_c, E)$
Spatial modes	Labelled vertices $\in U$
Creation operators	Unlabelled vertices $\in V_c$
Spatial distributions of creation operators	Edges $\in E$
Probability amplitude $\alpha_j^{(l)}$	Edge weight $\alpha_j^{(l)}$
Internal state $\in \{\|0\rangle, \|1\rangle\}$	Edge color $\in \{Blue, Red\}$
Final states w/ PS	Perfect matchings (PMs)

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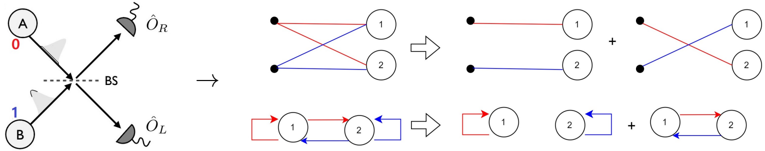
Example: $N = 2$ Bell state generation



Postselected schemes: PM digraph and separability

Definition

For a given G_d, we define a "perfect matching digraph" (PM digraph, $\overline{G}_d$) of the G_d as a directed subgraph in which only the loops and the edges included in the elementary cycles of the G_d are retained.

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Theorem 1

If an LQS generates a genuinely entangled no-bunching final state,
i) each vertex in the $\overline{G}_d$ must have more than two incoming edges of different colors
ii) all the vertices in it are strongly connected to each other
(strong connection of (w_i, w_j): we can move from w_i toward w_j and from w_j toward w_i)

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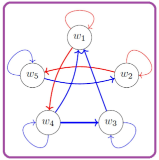
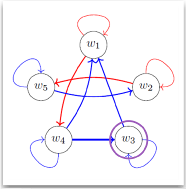
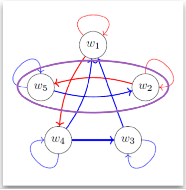


PM digraph

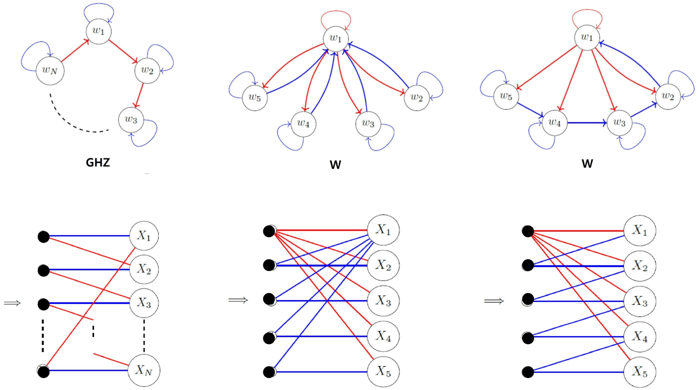




$$|\Psi_{fin}\rangle = \left[T_{11}T_{33}T_{44}|\downarrow_1\uparrow_4\rangle + T_{14}(T_{41}T_{33} + T_{43}T_{31})|\uparrow_1\downarrow_4\rangle\right] \otimes |\uparrow_3\rangle \otimes \left[T_{22}T_{55}|\downarrow_2\uparrow_5\rangle + T_{25}T_{52}|\uparrow_2\downarrow_5\rangle\right].$$

Postselected schemes: From $\overline{G}_d$ to genuinely entangled states

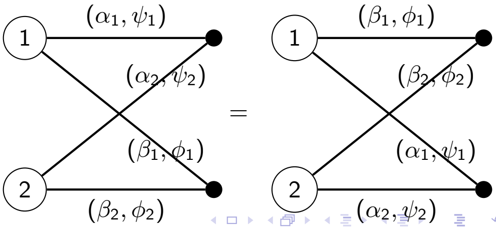
- The above theorem is powerful for designing an LQS for a specific genuinely entangled state
- Examples: GHZ, W, Dicke, Star network graph states, etc.



SC, Y.S. Kim, & M. Karczewski, arXiv:2211.04042

Boson systems with sculpting operators	**Bipartite Graph** $G_b = (U \cup V_a, E)$
Spatial modes	Labelled vertices $\in U$
$\hat{A}^{(l)}$ $(l \in \{1, 2, \cdots, N\})$	Unlabelled vertices $\in V_a$
Spatial distributions of $\hat{A}^{(l)}$	Edges $\in E$
Probability amplitude $\alpha_j^{(l)}$	Edge weight $\alpha_j^{(l)}$
Internal state $\psi_j^{(l)}$	Edge weight $\psi_j^{(l)}$

$$\hat{A}_2 = \hat{A}^{(2)}\hat{A}^{(1)} = (\beta_1 \hat{a}_{1,\phi_1} + \beta_2 \hat{a}_{2,\phi_2})(\alpha_1 \hat{a}_{1,\psi_1} + \alpha_2 \hat{a}_{2,\psi_2}) =$$



Property 1: No-bunching restriction in the graph picture
For a sculpting bigraph to $\hat{A}_N$, the probability amplitude weights are restricted so that only the perfect matchings (PMs) contribute to the final state $|\Psi\rangle_{fin} = \hat{A}_N|Sym_N\rangle$.

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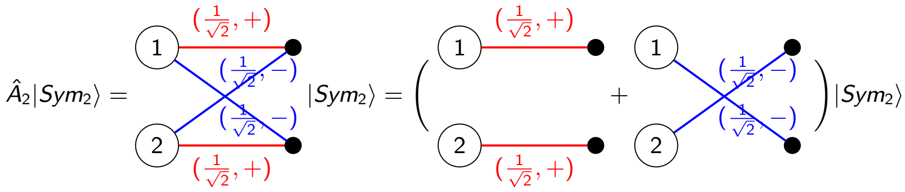
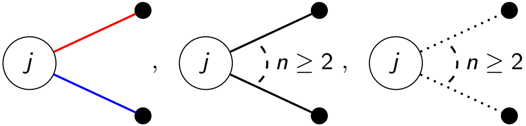
($N = 2$ example)
Setting $\hat{A}_2 = \hat{A}^{(1)}\hat{A}^{(2)} = \frac{(\hat{a}_{1+}+\hat{a}_{2-})}{\sqrt{2}}\frac{(\hat{a}_{2+}+\hat{a}_{1-})}{\sqrt{2}}$ $\quad (|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}})$,



$$= \frac{1}{2}(\hat{a}^\dagger_{1+}\hat{a}^\dagger_{2+} + \hat{a}^\dagger_{1-}\hat{a}^\dagger_{2-})|vac\rangle \qquad \text{(Bell state)}$$

(Black= $|0\rangle$, Dotted=$|1\rangle$, Red= $|+\rangle$, Blue= $|-\rangle$)

(Definition) An effective PM (EPM) bigraph is a bigraph whose edges always attach to the circles as one of the following forms:



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Property 2

If a sculpting operator is represented as an effective PM bigraph, then the final state is always fully determined by the PMs of the bigraph (the converse is not true)

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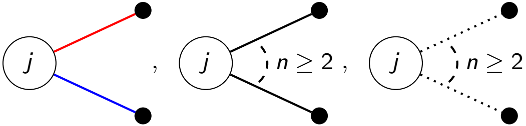

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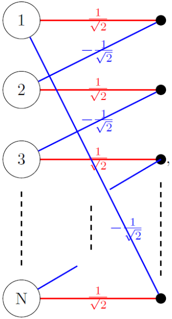
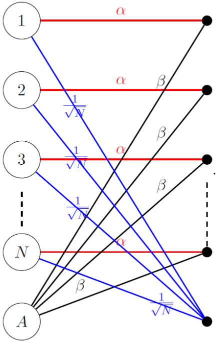
Theorem 2
If an EPM bigraph generates multipartite genuine entanglement, its $\bar{G}_d$ satisfies the two conditions of Thm 1

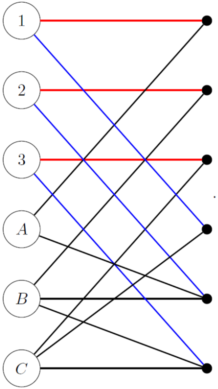
Sculpting schemes: From $\overline{G}_d$ to genuinely entangled states



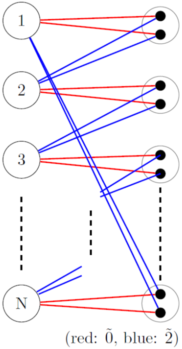
GHZ



W



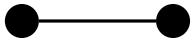
Superposition of N=3 GHZ and W



Qudit GHZ

SC, "From linear quantum system graphs to qubit graphs: Heralded generation of graph states", arXiv:2306.15148.

- Graph state: $G = (V, E) \leftrightarrow (\text{qubit}, U^Z)$
  where $U^Z|ij\rangle = (-1)^{ij}|ij\rangle$ (Controlled Z gate), e.g.,



$$= U^Z_{12}|++\rangle = |00\rangle + |01\rangle + |10\rangle - |11\rangle$$

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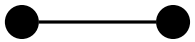

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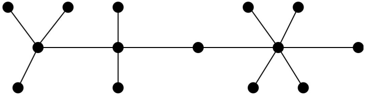
- Caterpillar graphs: Tree graphs in which every vertex is on a central path or one edge away from the path, e.g.,



Sculpting schemes: From LQS graphs to qubit graphs

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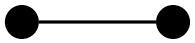

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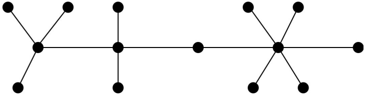
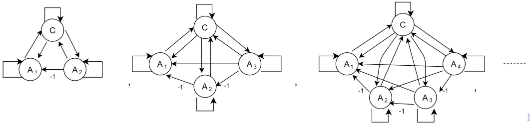
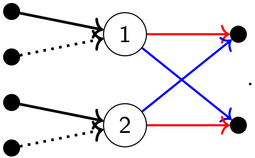


- We can generate any caterpillar graph state with EPM digraphs

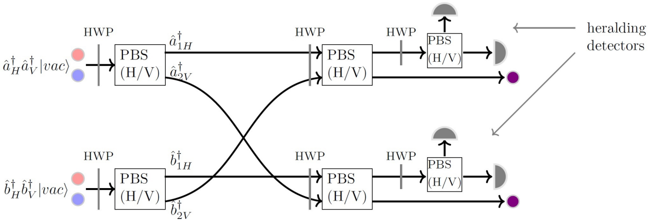


- $N = 2$ Bell state example

$$|\Psi\rangle_{fin} = \hat{A}_2|Sym_2\rangle =$$

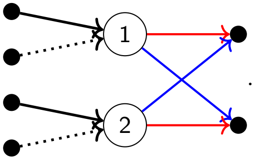


$$\Downarrow \quad \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\} \rightarrow \{|H\rangle, |V\rangle, |D\rangle, |A\rangle\}$$

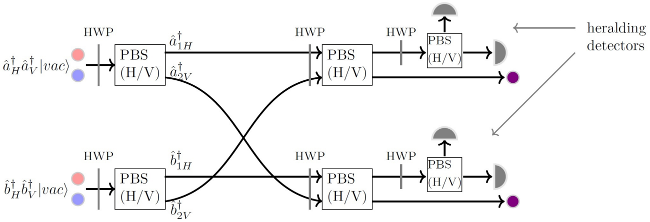


- $N = 2$ Bell state example

$$|\Psi\rangle_{fin} = \hat{A}_2|Sym_2\rangle = $$

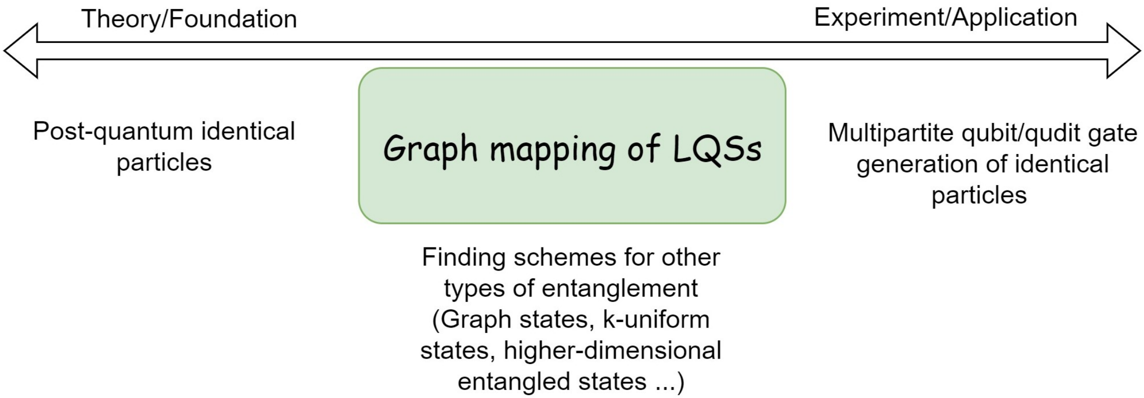


$$\Downarrow \quad \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\} \to \{|H\rangle, |V\rangle, |D\rangle, |A\rangle\}$$



- Transformation rules from a sculpting bigraph to an optical scheme (in preparation)

Discussions



- Physical heralded scheme designs + Experiments
- Sculpting schemes to generate other types of entanglement
- Directed bigraphs $\Longleftrightarrow$ XZ calculus
- Graph-based post-quantum theory of identical particles

Thank you!

- Chin, Seungbeom, Yong-Su Kim, and Sangmin Lee. "Graph picture of linear quantum networks and entanglement." Quantum 5 (2021): 611.
- Chin, Seungbeom, Yong-Su Kim, and Marcin Karczewski. "Shortcut to Multipartite Entanglement Generation: A Graph Approach to Boson Subtractions", arXiv preprint arXiv:2211.04042 (2022).
- Chin, Seungbeom, "From linear quantum system graphs to qubit graphs: Heralded generation of graph states", arXiv preprint, arXiv:2306.15148 (2023).