The Algebra for Stabilizer Codes

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July 21, 2023

ArXiv: 2304.10584

Outline

The ZX-calculus

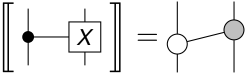
Graphical linear/affine algebra and the ZX-calculus

Graphical affine Lagrangian algebra and stabilizer circuits [Comfort and Kissinger, 2022]

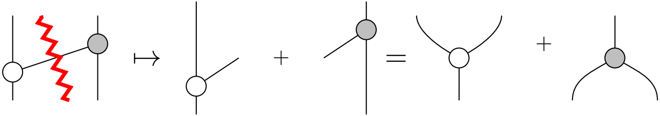
Graphical affine coisotropic algebra and stabilizer codes [Comfort, 2023]

The ZX-calculus “splits the atom”

The ZX-calculus decomposes quantum circuits into smaller components. Consider the controlled-X gate:



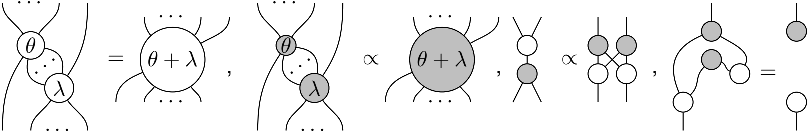
“ **Splitting the attom** ”

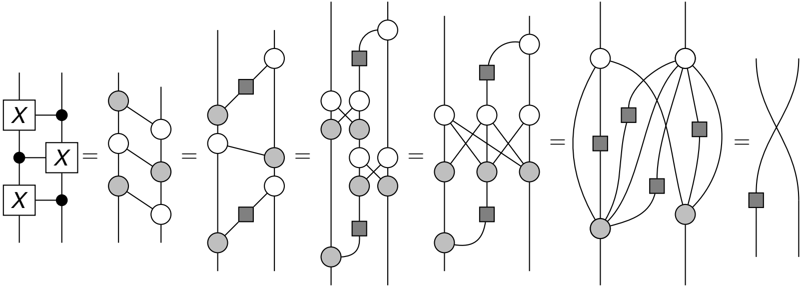


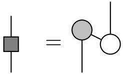
These components are no longer circuits, but they are useful.

Spider fusion and Hopf law

Spider fusion: **Bialgebra:** **Hopf law:**



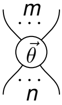


Where  is the antipode

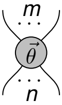
Interpreting the ZX-calculus

The d dimensional qudit (pure) **ZX-caculi** fare a family of graphical languages for $d^n \times d^m$ dimensional complex matrices.

There are two families of generators, Z and X spiders, decorated by phases $\vec{\theta} = (0, \theta_1, \theta_2, \ldots, \theta_{d-1}) \in [0, 2\pi)^d$:



$$\left[\!\left[\cdot \right]\!\right] = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i\theta_j} |j, \ldots, j\rangle\langle j, \ldots, j|$$



$$\left[\!\left[\cdot \right]\!\right] = \frac{1}{\sqrt{d}} \mathcal{F}^{\dagger} \sum_{j=0}^{d-1} e^{i\theta_j} |j, \ldots, j\rangle\langle j, \ldots, j| \mathcal{F}$$

Terminology:

qudit $\rightarrow$ **qubit** when $d = 2$.

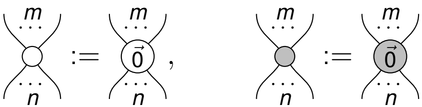
qudit $\rightarrow$ **quopit** when is an d odd prime.

Fragments of the ZX-calculus

Fragments of the ZX-calculus have restricted phases:

Example

The **qudit phase-free fragment** ZX-calculus has trivial angles:



The **qubit stabilizer fragment** has Z/X angles in:

$$\{(0,0),(0,\pi/2),(0,\pi),(0,2\pi/3)\} \subseteq [0,2\pi)^2$$

The **quopit stabilizer fragment** has Z/X angles in:

$$\left\{ \prod_{j=0}^{p-1}(n \cdot j + m \cdot j^2)\pi/p \;\middle|\; \forall n,m \in \mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z} \right\} \subseteq [0,2\pi)^p$$

Linear relations

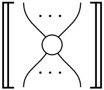
Definition

Given a field k there is a prop $\mathrm{LinRel_k}$ of **linear relations**, whose maps $n \to m$ are linear subspaces of $k^n \oplus k^m$ under relational composition: For $S \subseteq k^n \oplus k^m$ and $R \subseteq k^m \oplus k^\ell$

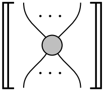
$$S; R := \{(x, z) \in k^n \oplus k^\ell \mid \exists y \in k^m : (x, y) \in S \wedge (y, z) \in R\}$$

Lemma

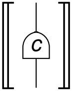
$\mathrm{LinRel_k}$ *is generated by two spiders, scalars (+ equations):*



$$= \left\{ \left(\begin{pmatrix} a \\ \vdots \\ a \end{pmatrix}, \begin{pmatrix} a \\ \vdots \\ a \end{pmatrix} \right) \,\middle|\, \forall a \in k \right\}$$



$$= \left\{ \left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \right) \,\middle|\, \forall a_i, b_j \in k : \sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j \right\}$$



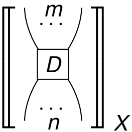
$$= \{(a, a \cdot c) \mid \forall a \in k\}$$

Lemma ([Zanasi, 2018])

$\mathrm{LinRel}_{\mathbb{F}_p}$ *is isomorphic to the p-dimensional qupit phase-free ZX-calculus modulo scalars.*

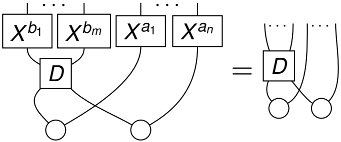
Proof.

A phase free ZX-diagram D is identified with its X stabilizers:



$:=$

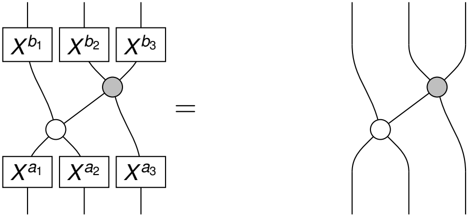
$$\left\{ \left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \right) \in \mathbb{F}_p^n \oplus \mathbb{F}_p^m \;\middle|\; \right.$$



$\}$ □

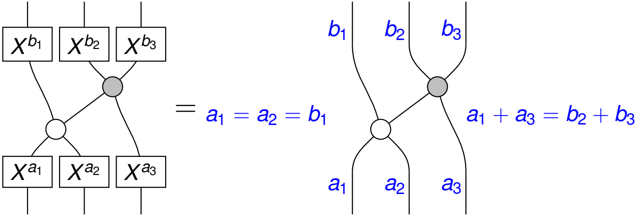
Example: part i

Find the $a_1, a_2, a_3, b_1, b_2, b_3$ which satisfy:

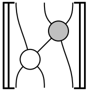


Example: part i

Find the $a_1, a_2, a_3, b_1, b_2, b_3$ which satisfy:



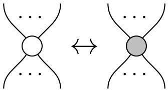
These equations determine a linear subspace of $\mathbb{F}_p^3 \oplus \mathbb{F}_p^3$:



$$\llbracket \cdot \rrbracket_X = \left\{ \left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right) \middle| a_1 = a_2 = b_1 \wedge a_1 + a_3 = b_2 + b_3 \right\}$$

What about the Z stabilizers?

The **Fourier transform** of a ZX-diagram is the colour swapping:



In the phase-free ZX-calculus, this corresponds to the **orthogonal complement** of linear subspaces:

$$(\_)^{\perp} : \mathsf{LinRel}_{\mathbb{F}_p} \to \mathsf{LinRel}_{\mathbb{F}_p};$$

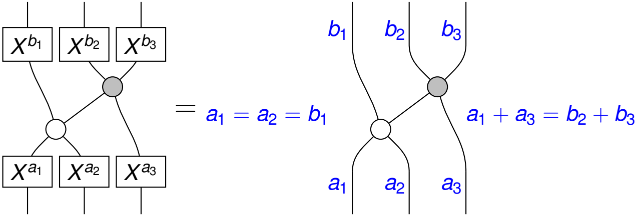
$$\left(S \subseteq \mathbb{F}_p^n\right) \mapsto \left(\{v \in \mathbb{F}_p^n | \forall w \in S, v^T w = 0\} \subseteq \mathbb{F}_p^n\right)$$

So we can also identify phase-free ZX-diagrams D with their Z stabilizers $[\![D]\!]_Z$:

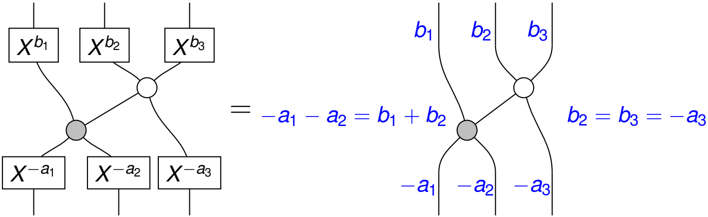
$$[\![D]\!]_Z = [\![D]\!]_X^{\perp}$$

Example: part ii

Recall how we calculated the X stabilizers:



For Z stabilizers, find X stabilizers of the Fourier transform:



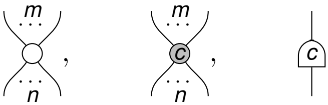
Affine relations

Definition ([Bonchi et al., 2019])

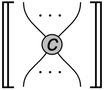
There is a prop AffRel_k of **affine relations**, but now the morphisms are (possibly empty) *affine* subspaces of $k^n \oplus k^m$.

Lemma

AffRel_k *is presented by decorating the grey spiders of* $\mathrm{LinRel_k}$ *with all elements $c \in k$:*



with the interpretation:

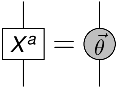


$$\left[\!\left[\cdots \right]\!\right] = \left\{ \left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \right) \middle| \; \forall a_i, b_j \in k : c + \sum a_i = \sum b_j \right\}$$

Modulo even more equations.

X-phase fragment of ZX and affine relations

The X gate is a phase:



, where $\vec{\theta} = (0a\pi/p, 1a\pi/p, 2a\pi/p, \ldots, (p-1)a\pi/p)$

So consider the **phase-free+X fragment** of the ZX-calculus.

Theorem

$\mathrm{AffRel}_{\mathbb{F}_p}$ *is isomorphic to the p-dimensional qudit phase free+X ZX-calculus gate modulo scalars.*

Proof.

The affine shift and X-gate have the same effect on X stabilizers.

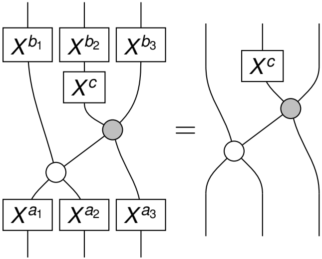
$$X^a = \sum_{x=0}^{p-1} |x+a\rangle\langle x|,$$

$\Big[\!\!\Big[$  $\Big]\!\!\Big] = \{(x, x+a)\}$

□

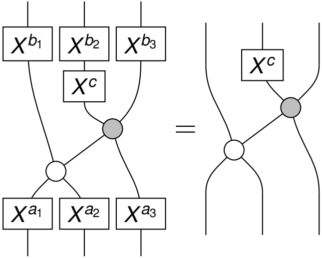
Example: part iii

What are the $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{F}_p$ which satisfy:

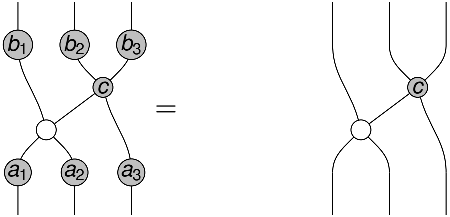


Example: part iii

What are the $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{F}_p$ which satisfy:

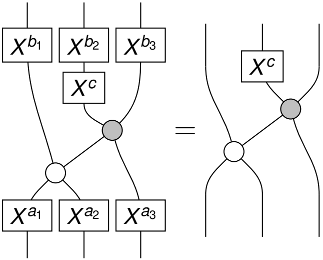


We can translate this into affine relations:



Example: part iii

What are the $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{F}_p$ which satisfy:



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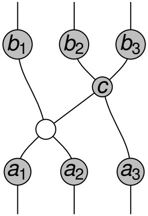
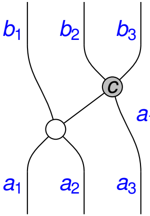


$= \; a_1 = a_2 = b_1$

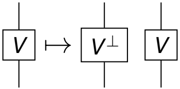


$a_1 + a_3 + c = b_2 + b_3$

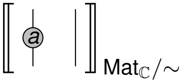
Adding Z and X gates to the picture

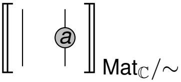
We can combine the Z and X-stabilizers in the same picture.

The phase-free circuits are doubled linear relations:

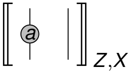


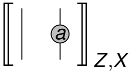
And add *both* the Z and X gates at once:

 $= Z^a$

 $= X^a$

The Z/X gates shift the Z/X stabilizers:

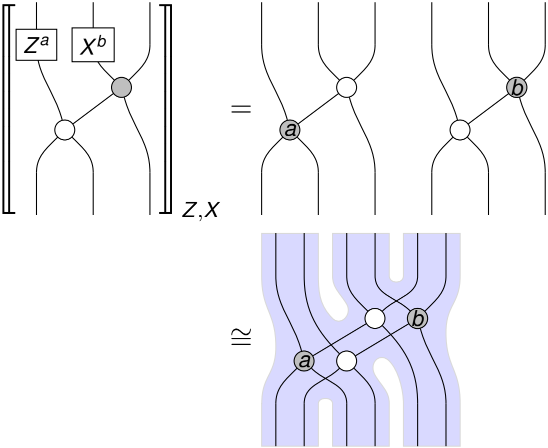
 $= \left\{ \left(\begin{pmatrix} z \\ x \end{pmatrix}, \begin{pmatrix} z+a \\ x \end{pmatrix} \right) \right\}$,

 $= \left\{ \left(\begin{pmatrix} z \\ x \end{pmatrix}, \begin{pmatrix} z \\ x+a \end{pmatrix} \right) \right\}$

This captures the **phase-free+Z+X** fragment of the ZX calculus

Example: part iv

Consider the interpretation of this ZX diagram:



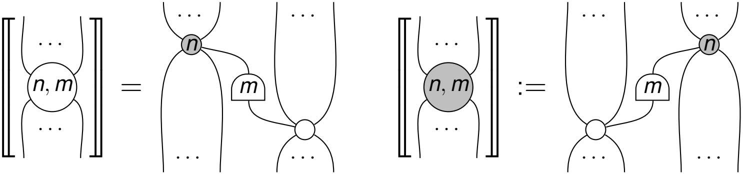
The Z and X stabilizers don't interact with each other for phase-free+Z+X circuits...

Getting all stabilizer circuits

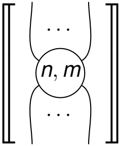
We can add the phase-shift gates to the picture:

Theorem ([Comfort and Kissinger, 2022])

Quopit stabilizer circuits are generated by the affine relations:



The white spider has the following interpretation in $\mathrm{Mat}_{\mathbb{C}}$*:*

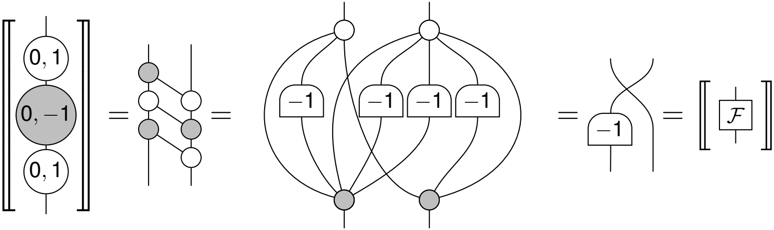


$$= \frac{1}{\sqrt{p}} \sum_{j=0}^{p-1} e^{\pi \cdot i (n \cdot j + m \cdot j^2)/p} |j, \ldots, j\rangle\langle j, \ldots, j|$$

The interpretation of the grey spider is analogous.

“Splitting the atom” $\times 2$

The Fourier transform is derived by Euler composition:



And the Euler decomposition is derived from the Hopf law!

Symplectic algebra and Weyl operators

Definition

Given $(z_1, \ldots, z_n, x_1, \ldots, x_n) \in \mathbb{F}_p^{2n}$, there is a **Weyl operator**:

$$\mathcal{W}(z, x) := \bigotimes_{j=1}^{n} Z_{(j)}^{z_j} X_{(j)}^{x_j} : \mathbb{C}^n \to \mathbb{C}^n$$

The **symplectic form** $\omega : \mathbb{F}_p^{2n} \oplus \mathbb{F}_p^{2n} \to \mathbb{F}_p$ takes

$$\left(\begin{pmatrix} z \\ x \end{pmatrix}, \begin{pmatrix} z' \\ x' \end{pmatrix} \right) \mapsto z^T x' - x'^T z$$

This form captures the commutation of Weyl operators:

$$\mathcal{W}(z, x)\mathcal{W}(z', x') = e^{i \cdot \pi \cdot \omega((z,x),(z',x'))/p} \mathcal{W}(z', x')\mathcal{W}(z, x)$$

Stabilizer circuits are affine Lagrangian relations

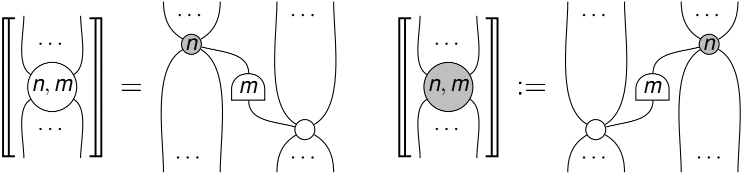
Definition

Lagrangian subspaces are linear subspaces $V \subseteq \mathbb{F}_p^{2n}$ st:

$$V = V^\omega := \{v \in \mathbb{F}_p^{2n} | \forall w \in V, \omega(v, w) = 0\}$$

These are linear subspaces of $\mathbb{F}_p^{2n}$ where all elements commute wrt the symplectic form.

The two spiders we mentioned actually generate the prop $\mathrm{AffLagRel}_{\mathbb{F}_p}$ of **affine Lagrangian relations** over $\mathbb{F}_p$:

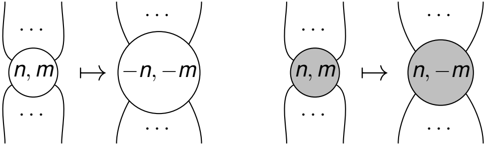


The view of stabilizer *states* in terms of affine Lagrangian subspaces acted on by reversible transformations with measurement statistics was shown in [Calderbank et al., 1998, Gross, 2006, Catani and Browne, 2017, De Beaudrap, 2013]

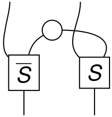
Mixed stabilizers

The complex conjugation is given by

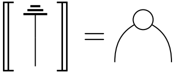
$$\overline{(\_)} : \mathrm{AffLagRel}_{\mathbb{F}_p} \to \mathrm{AffLagRel}_{\mathbb{F}_p};$$



Mixed stabilizer circuits are constructed by doubling stabilizer circuits S:



The quantum discard is the cap:



Formally this is taking
$\mathrm{CPM}(\mathrm{AffLagRel}_{\mathbb{F}_p}, \overline{(\_)}) \cong \mathrm{CPM}(\mathrm{Stab}_p, \overline{(\_)})/\sim$

Stabilizer codes

Definition

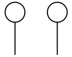
There is a category of **affine coisotropic relations** $\mathsf{AffCoIsotRel}_{\mathbb{F}_p}$ whose morphisms $n \to m$ are affine subspaces

$$L + a \subseteq \mathbb{F}_p^{2(n+m)} \text{ such that } L^\omega \subseteq L$$

(relaxed from affine Lagrangian subspaces where $L^\omega = L$)

Proposition ([Comfort, 2023])

$\mathsf{AffCoIsotRel}_{\mathbb{F}_p}$ *is presented by adding the discard* relation *to* $\mathsf{AffLagRel}_{\mathbb{F}_p}$*:*

$[\![$  $]\!]$ $= \left\{ \left(\begin{pmatrix} z \\ x \end{pmatrix}, * \right) \,\middle|\, \forall z, x \in \mathbb{F}_p \right\}$

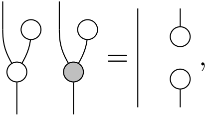
Theorem ([Comfort, 2023])

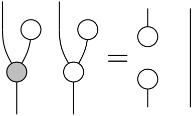
$$\mathsf{AffCoIsotRel}_{\mathbb{F}_p} \cong \mathsf{CPM}(\mathsf{AffLagRel}_{\mathbb{F}_p}) \cong \mathsf{CPM}(\mathsf{Stab}_p)/\sim$$

The quantum discard is the discard relation!
(reminiscent of [Carette et al., 2021])

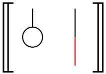
Measurement

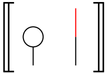
The Z and X projectors look as follows:

$p_Z :=$  ,

$p_X :=$ 

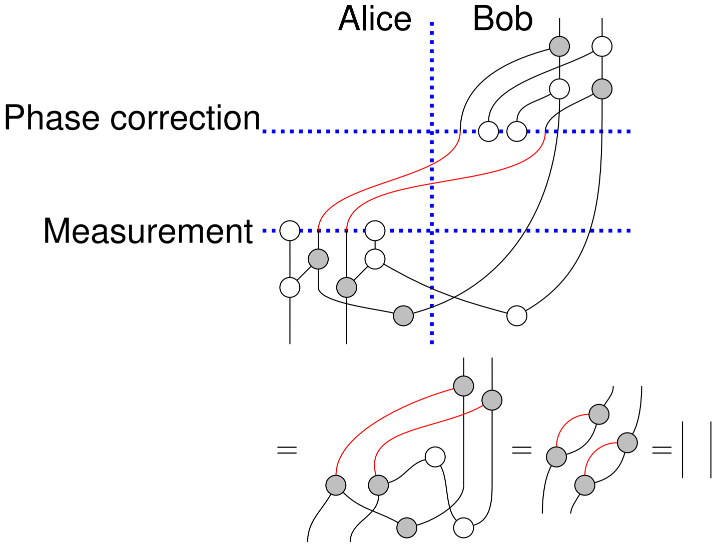
By splitting the X projector, following [Selinger, 2008], we get state preparation and measurement relations:

 $= \left\{ \left(x, \begin{pmatrix} z \\ x \end{pmatrix} \right) \middle| \forall z, x \in \mathbb{F}_p \right\}$

 $= \left\{ \left(\begin{pmatrix} z \\ x \end{pmatrix}, x \right) \middle| \forall z, x \in \mathbb{F}_p \right\}$

Possible outcomes of Pauli measurements of stabilizer states are equally likely.

Quantum teleportation by spider fusion



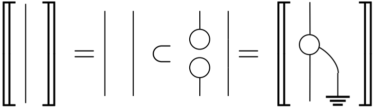
(this also works for qubits)

Maps between stabilizer codes

This is a 2-category with respect to subspace inclusion:

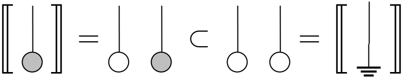
Example

All stabilizers for the identity channel are stabilizers for the Z/X-projectors:



Example

All stabilizers for $|0\rangle$ are stabilizers for the maximally mixed state:

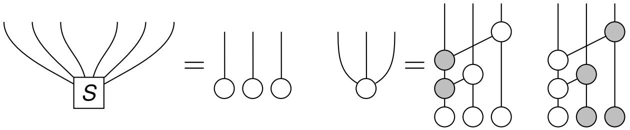


Error correction example: repetition code

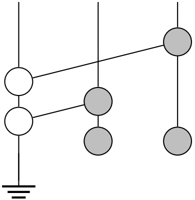
Consider the Linear subspace:

$$S = \{((z_1, z_2, z_3), (x_1, x_2, x_3)) | x_1 = x_2 = x_3\} \subseteq \mathbb{F}_2^{2(3)}$$

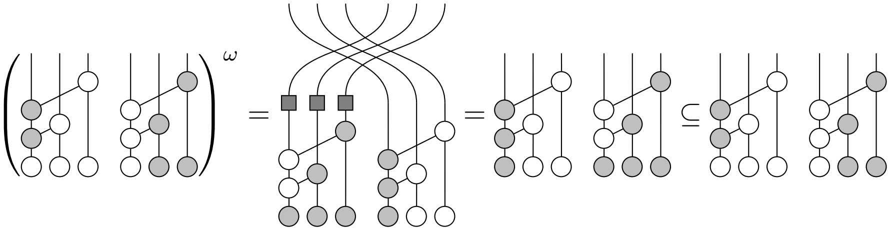
Which can be written in the form of a circuit:



In the undoubled picture this looks like:

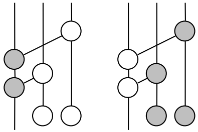


Recall S is coisotropic if $S^{\omega} \subseteq S$:



Moreover $S \subseteq \mathbb{F}_2^{2(3)}$ has dimension $3 + 1 = 4$

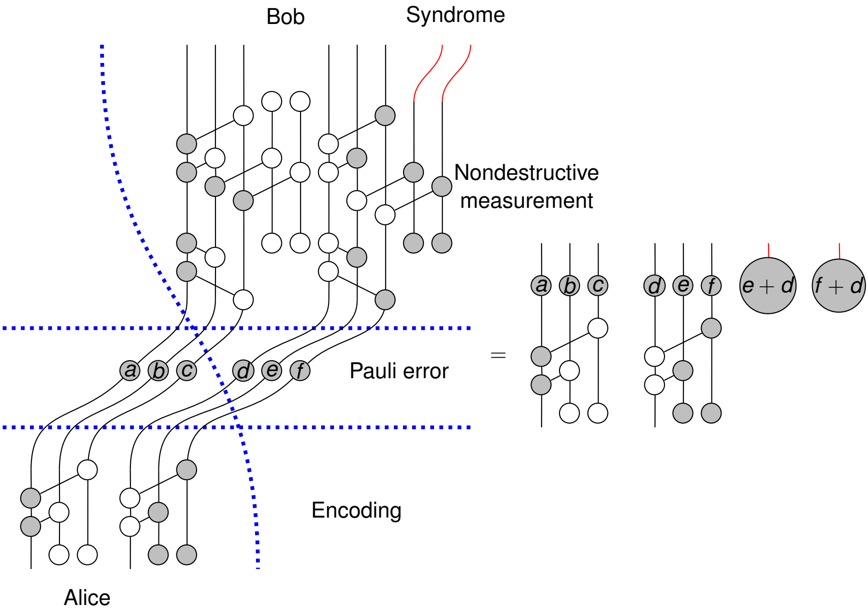
Chopping off the maximally mixed state gives us an encoding map $\mathbb{F}_2^{2(1)} \to \mathbb{F}_2^{2(3)}$:



That is, this is a qubit $[3, 1]$-stabilizer code.

Encodes 1 logical qubit into 3 physical qubits.

Suppose there is a Pauli error $W((a, b, c), (d, e, f))$, then we have the following error detection circuit:



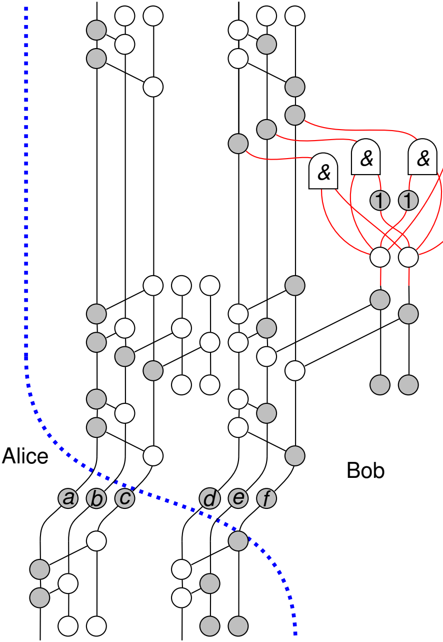
Suppose we want to correct for single pauli X errors, then we find that:

$$X \text{ error } (d,e,f) = (1,0,0) \text{ syndrome } (e+d, f+d) = (1,1)$$
$$X \text{ error } (d,e,f) = (0,1,0) \text{ syndrome } (e+d, f+d) = (1,0)$$
$$X \text{ error}(d,e,f) = (0,0,1) \text{ syndrome } (e+d, f+d) = (0,1)$$
$$X \text{ error } (d,e,f) = (0,0,0) \text{ syndrome } (e+d, f+d) = (0,0)$$

Therefore, we want to apply the correction

$$\mathbb{F}_2^2 \to \mathbb{F}_2^{2(3)}; (s,t) \mapsto (\ (0,0,0), (st, s(t+1), (s+1)t)\ )$$

The error correction protocol then has the following form:



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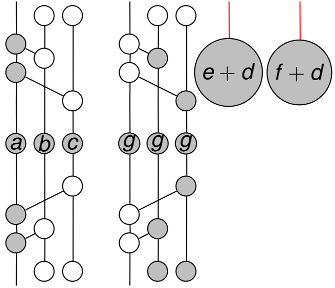


where $g := d + (e + d)(f + d)$

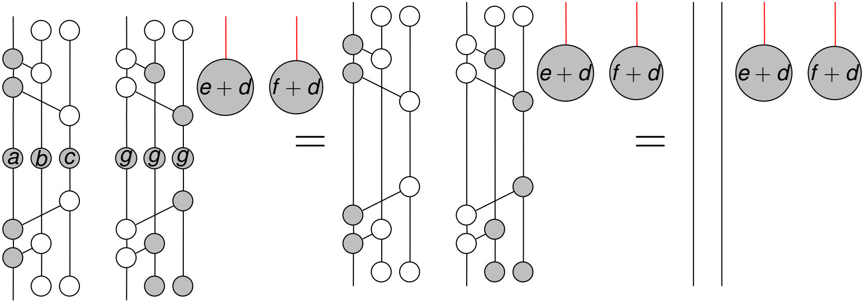
$$= e + (e + d)(f + d + 1)$$

$$= f + (e + d + 1)(f + d)$$

$$= de + ef + fd$$

If there is at most one X error, then $g = de + ef + fd = 0$.

If furthermore, there are no Z errors:



Problems

The and gate isn't an affine subspace, so it doesn't live in affine coisotropic relations.

Using this formalism we can only *correct* for errors with affine post-processing.

This works for all odd prime qudit stabilizer codes, but only phase-free qubit stabilizer codes (CSS codes)+Z+X gates (no Fourier transform of phase shift)

Future work

Find nice semantics for stabilizer codes with nonlinear classical post-processing.

Undoubled completeness for arbitrary field (following [Booth and Carette, 2022, Poór et al., 2023])

Extending this work to optics/electrical circuits...

Code distance.

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