Indeterminism and Bell nonlocality with classical systems

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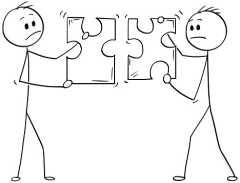
Overview

Determinism in classical theory

Classical theory has always been regarded as **deterministic**



Quantum Theory offers a new perspective

$$\rho_{\text{universe}} = |\Psi\rangle \langle\Psi|$$

whereas the system is individually in a mixed state[1]

While our toy theory is NOT meant to be a description of the real world, it shows that

$$\boxed{\text{classical physics formalism} \;\not\Longrightarrow\; \text{fundamental determinism}}$$

[1] E. Schrödinger, Math. Proc. Cambridge Phil. Soc. **31** 555 (1935)

Framework

Language of QT with Hilbert spaces in finite dimension:

- Dirac notation for pure states;
- density matrix formalism for mixed states;
- transformations are linear maps between (operator on) Hilbert spaces that preserve the state space structure;
- Measurements are described by positive-operator-valued measures (POVMs)

Classical probability theory

- all the pure states of every system are perfectly distinguishable through a single measurement;
- the pure states of a composite system are the products of pure states of the component systems;
- all permutations of the set of pure states are valid physical transformations.

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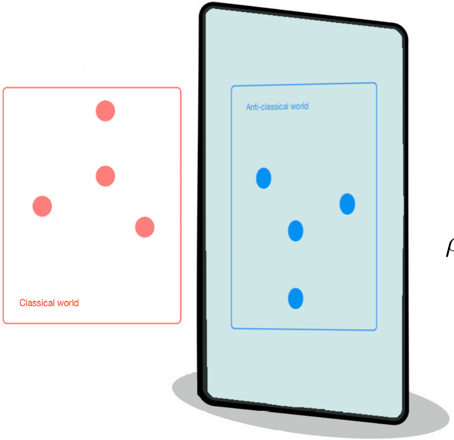
We will distinguish **two** types of classical systems:
classical systems and **anti-classical systems** and set up a rule about their composition.

Classical systems in a mirror

bit

$$\rho_{\mathrm{B}} = \alpha \left|0\right\rangle \left\langle 0\right| + \beta \left|1\right\rangle \left\langle 1\right|$$

$$\rho_{\mathrm{B}^{\otimes 2}} = \sum_{i,j=0}^{1} \alpha_{ij} \left|ij\right\rangle \left\langle ij\right|$$

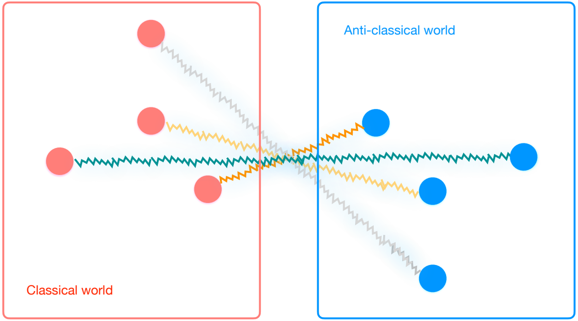


anti-bit

$$\rho_{\mathrm{A}} = \gamma \left|0\right\rangle \left\langle 0\right| + \delta \left|1\right\rangle \left\langle 1\right|$$

$$\rho_{\mathrm{A}^{\otimes 2}} = \sum_{i,j=0}^{1} \gamma_{ij} \left|ij\right\rangle \left\langle ij\right|$$

Entanglement as a yield of the joint system



bit⊗anti-bit

$$|\Psi\rangle_{\mathrm{B}\otimes\mathrm{A}} = \alpha\,|0\rangle_{\mathrm{B}}\,|0\rangle_{\mathrm{A}} + \beta\,|1\rangle_{\mathrm{B}}\,|1\rangle_{\mathrm{A}}$$

$$|\Phi\rangle_{\mathrm{B}\otimes\mathrm{A}} = \gamma\,|0\rangle_{\mathrm{B}}\,|1\rangle_{\mathrm{A}} + \delta\,|1\rangle_{\mathrm{B}}\,|0\rangle_{\mathrm{A}}$$

$$\rho = c_1\,|\Psi\rangle\,\langle\Psi| + c_2\,|\Phi\rangle\,\langle\Phi|$$

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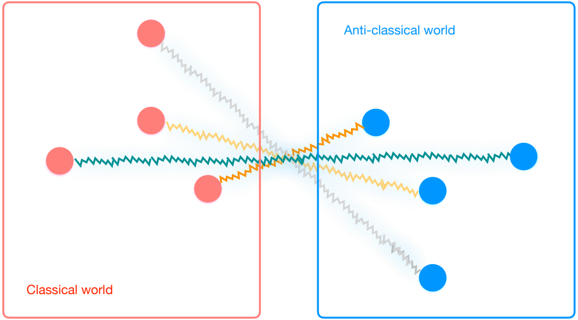

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$$|\Psi\rangle_{B\otimes A} = \alpha\,|0\rangle_B\,|0\rangle_A + \beta\,|1\rangle_B\,|1\rangle_A$$

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VIOLATION local tomography[2]

[2] G.M. D'Ariano, M. Erba, P. Perinotti, Phys. Rev A **101** (2020)

Why this state space structure?

Our starting point was to have **classical states** when considering **bit** or **anti-bit** systems **alone**.

If, for example, we consider

$$|\varphi\rangle_{B\otimes A} = \alpha\,|0\rangle_B\,|1\rangle_A + \beta\,|1\rangle_B\,|1\rangle_A \xrightarrow{\text{Trancing on A}} \sigma_\varphi = (\alpha\,|0\rangle_B + \beta\,|1\rangle_B)(\alpha^*\,\langle 0|_B + \beta^*\,\langle 1|_B)$$

Hence, in order to have a consistent theory, NOT all *quantum* states can be considered here!

Generalizing to dimension d $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$

bit $\rightarrow$ **dit**

anti-bit $\rightarrow$ **anti-dit**

A **pure** state of the composite of m **dits** and n $(> m)$ **anti-dits** is any vector

$$|\Psi\rangle_{D^{\otimes m}A^{\otimes n}} = \left[U^{(D_1 \ldots D_m)} \otimes W^{(A_1 \ldots A_n)}\right] \left(|\Psi'\rangle_{D_1 \otimes \cdots \otimes D_m \otimes A_1 \otimes \cdots \otimes A_m} \otimes |\mathbf{r}\rangle_{A_{m+1} \otimes \cdots \otimes A_n}\right)$$

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$|\mathbf{r}\rangle = |r_1\rangle \otimes \cdots \otimes |r_{n-m}\rangle \in \{0, \ldots, d-1\}^{\times (n-m)}$ is a product state

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$$|\Psi'\rangle = \left(\Pi_{k_1}^{D_1 A_1} \otimes \cdots \otimes \Pi_{k_m}^{D_m A_m}\right) |\Psi'\rangle ,$$

where $\Pi_{k_i}^{D_i A_i}$ is the projector onto the subspace $\mathrm{Span}\{|j\rangle\, |j \oplus k_i\rangle \mid j = 0, \ldots, d-1\}$ of the composite system of the i-th **dit** and i-th **anti-dit** with parity $k_i \in \{0, \ldots, d-1\}$.

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$U^{(D_1 \ldots D_m)}$ $(W^{(A_1 \ldots A_n)})$ is a permutation over dits (anti-dits)

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Mixed states are probability mixtures of pure states (also with different *parity*)

Measurements

Measurements on composite systems are described by POVMs $\{P_i\}$ whose operators are linear combinations, with positive coefficients, of allowed states.

[3] H. Barnum, C.P. Gaebler, A. Wilce, Found. Phys., **43** (2013)

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Uniform universal steering[3]

For every system S, there exist a system X, such that for every state $\rho \in S$, there exists a state $\omega \in SX$ that steers its marginal ρ on system S.

[3] H. Barnum, C.P. Gaebler, A. Wilce, Found. Phys., **43** (2013)

Reversible transformations

- **Permutations** acting on dit systems and anti-dit systems separately
- **Generalized bit flip**: $X : |j\rangle \mapsto |j \oplus 1\rangle \quad \& \quad X^{\mathbf{s}} : |\mathbf{j}\rangle \mapsto |j_1 \oplus s_1\rangle \otimes \ldots \otimes |j_n \oplus s_n\rangle$
- **Generalized phase flip**: $Z\,|j\rangle = \omega^j\,|j\rangle\,, \quad \omega = \exp\left\{\frac{2\pi i}{d}\right\}$

$$\left(\sigma_1 \otimes \sigma_2\right) \circ \left(X^{\mathbf{s}_1} \otimes X^{\mathbf{s}_2}\right) \circ \left(Z^{\mathbf{v}_1} \otimes Z^{\mathbf{v}_2}\right)$$

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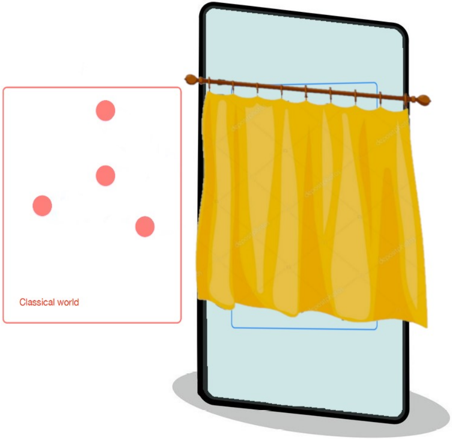
- **Dynamical evolution**: physical transformations $\mathcal{T}$ of type $A \to B$,

$$\rho \mapsto \mathcal{T}(\rho) := \mathrm{Tr}_{AC}[(\rho_A \otimes \sigma_{CB})\,(P_{AC} \otimes I_B)]$$

Classical theory is not a restriction, it is included!



- when looking at **classical** systems ONLY, we just have Classical Probability Theory
- only states, effects, and transformations of Classical Theory are admissible in the composition of **classical** systems
- hence Classical Probability Theory is included, in fact **twice**!

Entangled states really exist

SEPARABLE state

$$\rho_{B\otimes A} = p\,|00\rangle\,\langle 00| + (1-p)\,|11\rangle\,\langle 11|$$

ENTANGLED state

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Can they be distinguished? **Yes**, by performing...

POVM: $\{P_{\text{yes}}, P_{\text{no}}\}$ where $P_{\text{yes}} = |\Psi\rangle\langle\Psi|$ and $P_{\text{no}} = I \otimes I - P_{\text{yes}}$

there exist a strategy to **distinguish** between any separable and entangled state by repeatedly performing measurements on multiple copies of the target states

Purificable states

1. For every classical state, *i.e.* $\forall \rho \in \mathsf{St}(\mathrm{B}/\mathrm{A}^{\otimes M})$, there exists a purification.

For example, $\rho_B = |0\rangle\langle 0| + |1\rangle\langle 1|$ is purified by $|\Psi\rangle_{B\otimes A} = |0\rangle_B |0\rangle_A + |1\rangle_B |1\rangle_A$.

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2. However, NOT every state can be purified.

In $d = 3$, the mixture

$$\begin{aligned}\rho_{\mathrm{D}\otimes\mathrm{A}} =& p_1(\alpha^{(1)}|00\rangle + \beta^{(1)}|22\rangle)(\alpha^{(1)*}\langle 00| + \beta^{(1)*}\langle 22|)\\ &+p_2(\alpha^{(2)}|01\rangle + \beta^{(2)}|20\rangle)(\alpha^{(2)*}\langle 01| + \beta^{(2)*}\langle 20|)\\ &+p_3(\alpha^{(3)}|02\rangle + \beta^{(3)}|21\rangle)(\alpha^{(3)*}\langle 02| + \beta^{(3)*}\langle 21|)\end{aligned}$$

does not have any purification.

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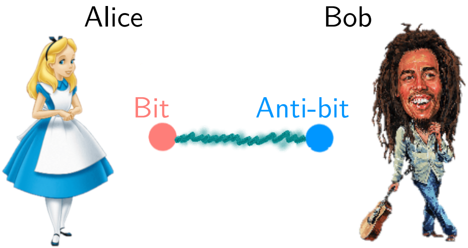
$\Rightarrow$ classical systems are **not** fundamentally deterministic!

Activation of Bell nonlocality

standard CHSH settings



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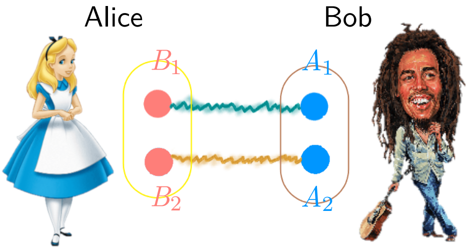

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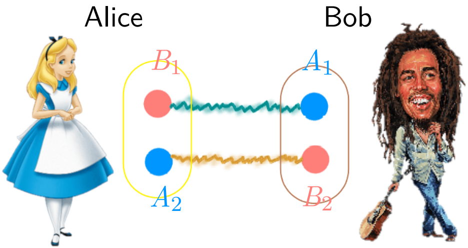

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- Bell nonlocality is activated when two copies of the state are considered
- For every pure entangled state, a **violation of CHSH** is achieved given two copies

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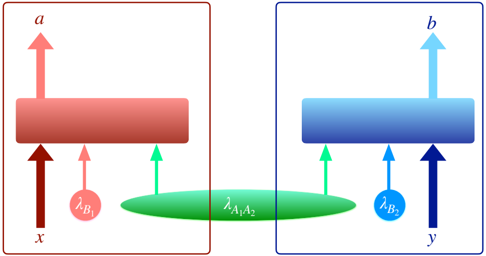

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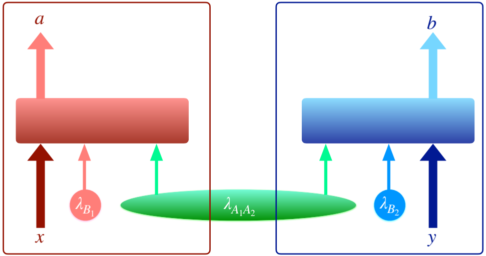

hence the violation of the CHSH inequality for a given ontic state λ of this form would imply the violation of the CHSH inequality for the state $\lambda_{A_1 A_2}$!

Conclusion

We introduced a toy theory in which every classical system can be entangled with a dual, anti-classical system.

The theory

- 1-to-1 correspondence between states and effects;
- can purify every classical state;
- exhibits the activation of Bell nonlocality.

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Thank you for your attention.