Light-matter interaction in the ZXW calculus

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Quantinuum

https://arxiv.org/abs/2306.02114

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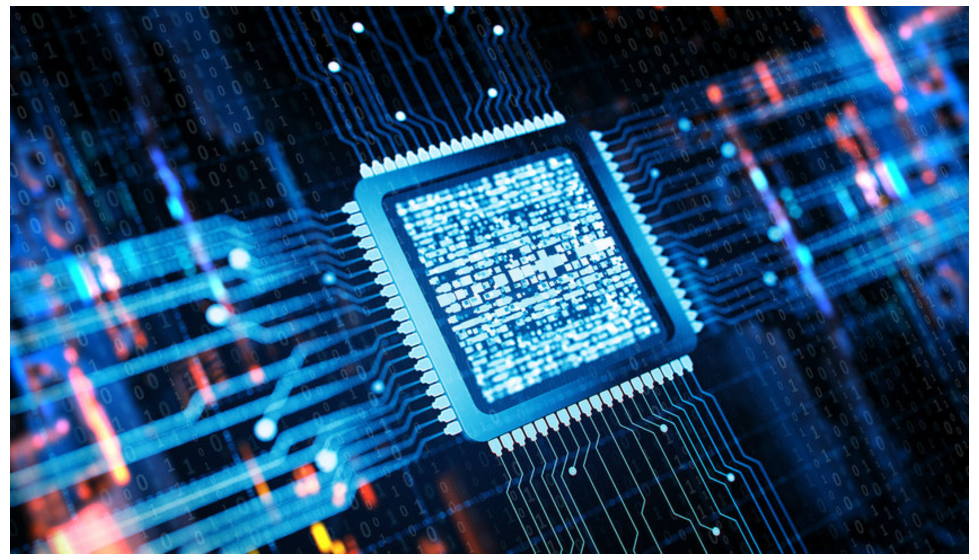
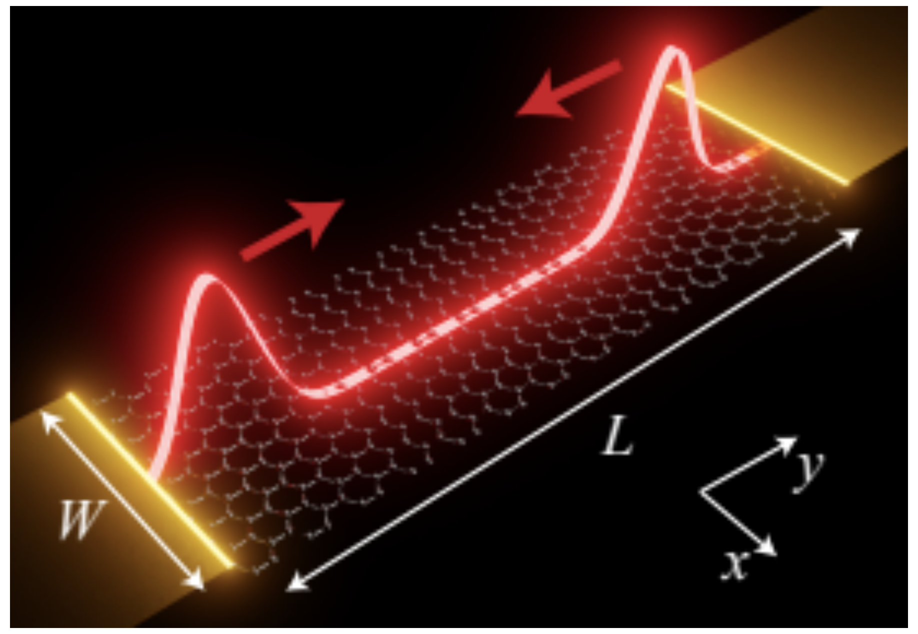
Motivation: photonic quantum computing

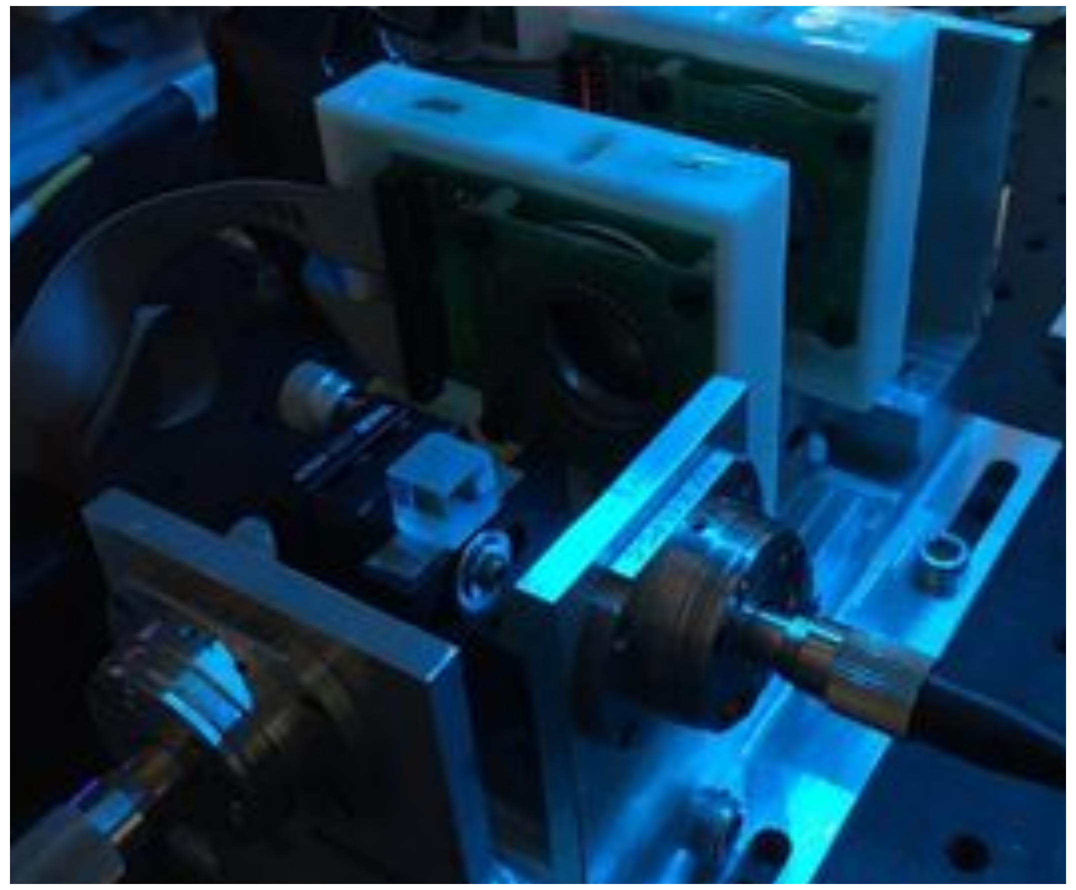
Photonic chip



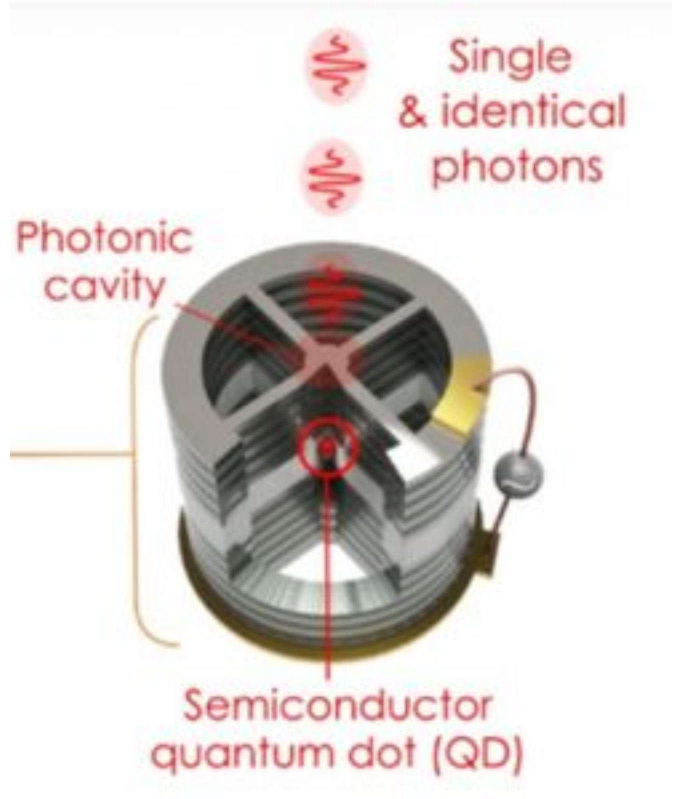
Kerr medium



Beam splitter



Quantum dot



The maths they use…

$$\hat{U}(t) = e^{-i\hat{H}_{2C}t/\hbar}$$
$$= \begin{pmatrix} e^{-i\omega_c t\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)}\left(\cos t\sqrt{\hat{\varphi}+g^2} - i\delta/2\frac{\sin t\sqrt{\hat{\varphi}+g^2}}{\sqrt{\hat{\varphi}+g^2}}\right) & -ige^{-i\omega_c t\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)}\frac{\sin t\sqrt{\hat{\varphi}+g^2}}{\sqrt{\hat{\varphi}+g^2}}\,\hat{a} \\ -ige^{-i\omega_c t\left(\hat{a}^\dagger\hat{a}-\frac{1}{2}\right)}\frac{\sin t\sqrt{\hat{\varphi}}}{\sqrt{\hat{\varphi}}}\hat{a}^\dagger & e^{-i\omega_c t\left(\hat{a}^\dagger\hat{a}-\frac{1}{2}\right)}\left(\cos t\sqrt{\hat{\varphi}} + i\delta/2\frac{\sin t\sqrt{\hat{\varphi}}}{\sqrt{\hat{\varphi}}}\right) \end{pmatrix}$$

- Quantization of electromagnetic field
- Creation and annihilation operators
- Hamiltonians

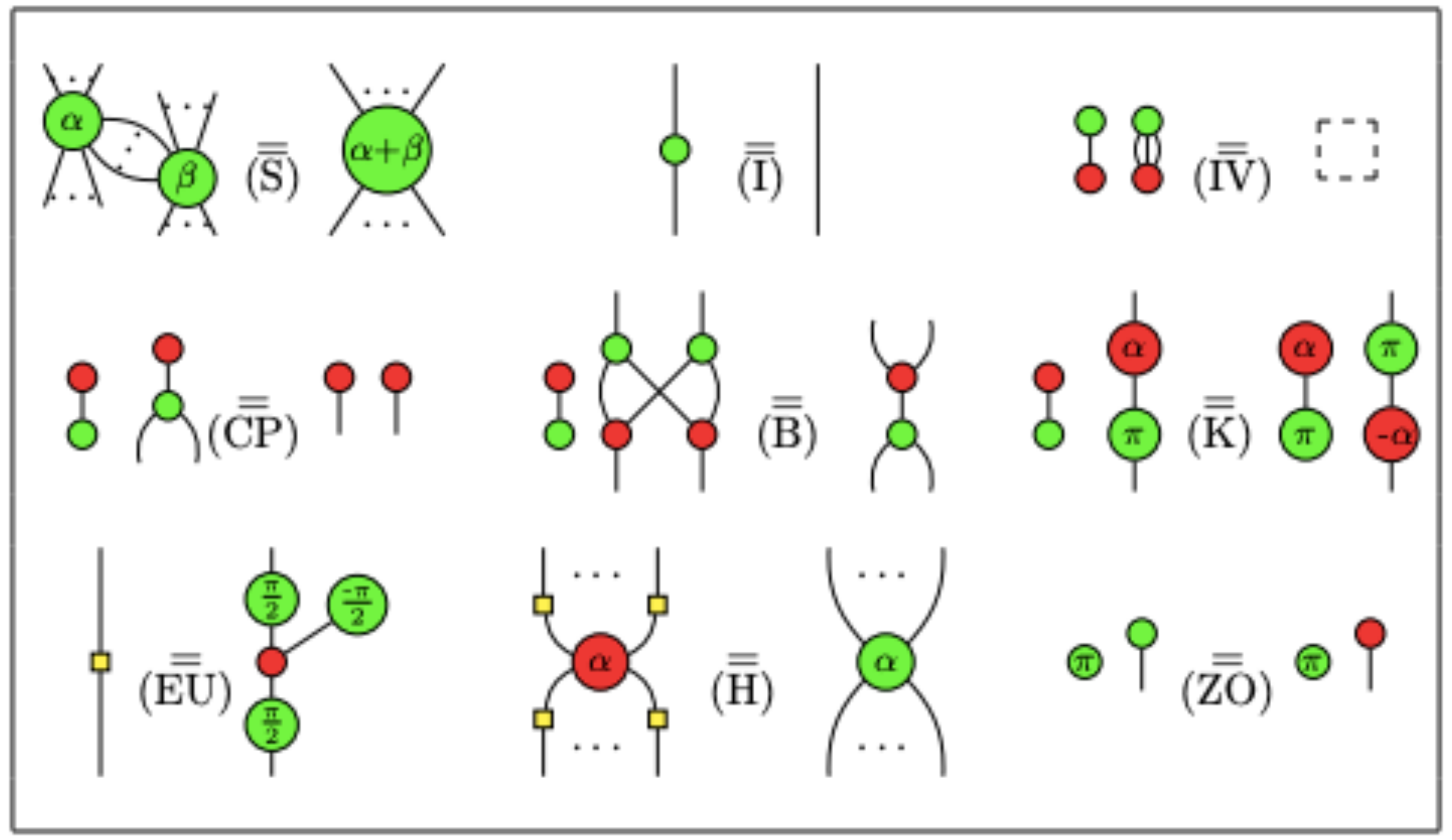
$$\frac{1}{\pi u}\exp\left(-\frac{\alpha\alpha^*}{u}\right) * |\alpha, m\rangle\langle\alpha, n|$$
$$= \frac{(-1)^{m+n}}{1+u}\sqrt{m!n!}\sum_{j=-\{m,n\}}^{\infty}[(m+j)!(n+j)!]^{1/2}|\alpha, m+j\rangle\langle\alpha, n+j|$$
$$\times \sum_{p=0}^{\{m,m+j\}}\sum_{q=0}^{\{n,n+j\}}\frac{(-1)^{p+q}(m+n+j-p-q)!}{p!(m-p)!(m+j-p)!q!(n-q)!(n+j-q)!}$$
$$\times\left(\frac{u}{1+u}\right)^{m+n+j-p-q}. \qquad \text{(B.1)}$$

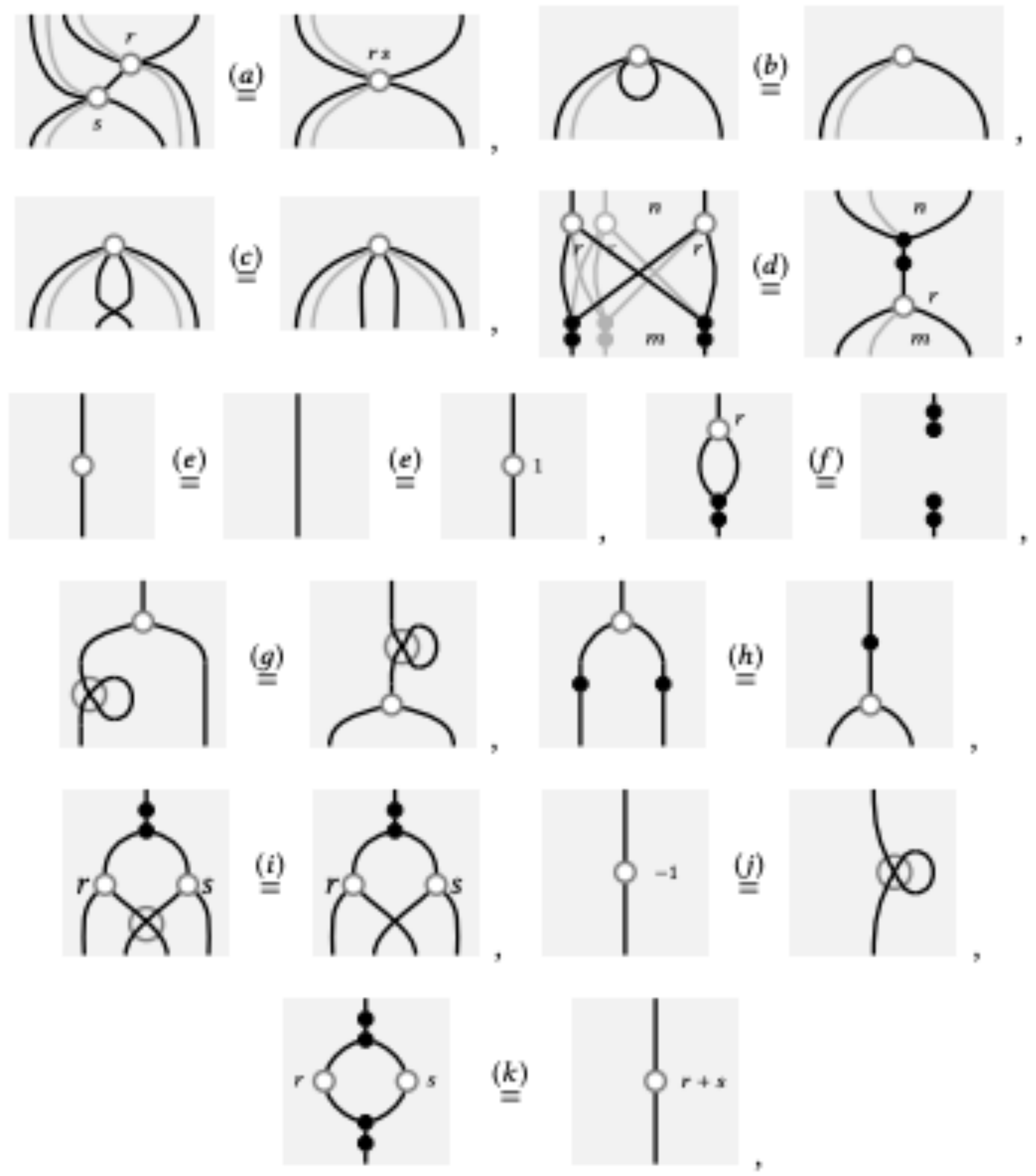
The double sum in equation (B.1) over p and q can be simplified in the following way ($x \equiv u/(1+u)$)

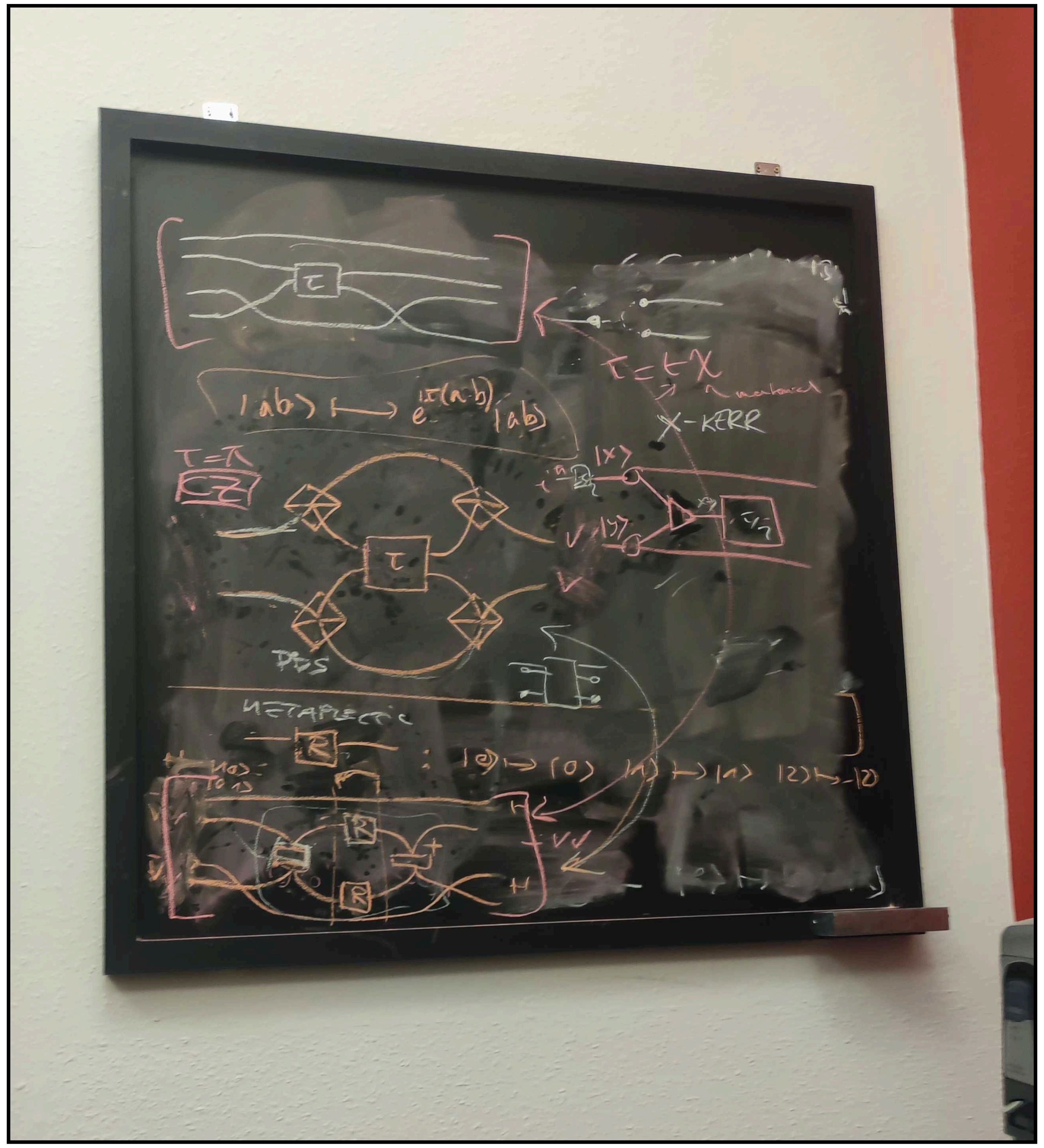
$$\sum_{p=0}^{\{m,m+j\}}\sum_{q=0}^{\{n,n+j\}}\frac{(-1)^{p+q}(m+n+j-p-q)!}{p!(m-p)!(m+j-p)!q!(n-q)!(n+j-q)!}x^{m+n+j-p-q}$$
$$= \frac{1}{n!}\sum_{p=0}^{\{m,n+j\}}\frac{(-1)^p x^{m-p}}{p!(m-p)!(m+j-p)!}\frac{\partial^{m-p}}{\partial x^{m-p}}\{x^{m+j-p}(x-1)^n\}$$
$$= \sum_{p=0}^{\{m,m+j\}}\sum_{k=0}^{m-p}\frac{(-1)^p x^{m-p+j+k}(x-1)^{n-k}}{p!k!(m-p-k)!(j+k)!(n-k)!} \qquad \text{(B.2)}$$
$$= x^{m+j}(x-1)^n\sum_{k=0}^{n}\frac{1}{k!(j+k)!(n-k)!}\left(\frac{x}{x-1}\right)^k\sum_{p=0}^{m-k}\frac{(-1)^p}{p!(m-k-p)!}\left(\frac{1}{x}\right)^p$$
$$= x^j(x-1)^{m+n}\sum_{k=0}^{\{m,n\}}\frac{1}{k!(j+k)!(m-k)!(n-k)!}\left(\frac{x}{x-1}\right)^{2k}.$$

In equation (B.2), we applied the binomial formula, Leibniz's rule for multiple differentiations of a product and a transposition of the summations. Substituting

The maths we want





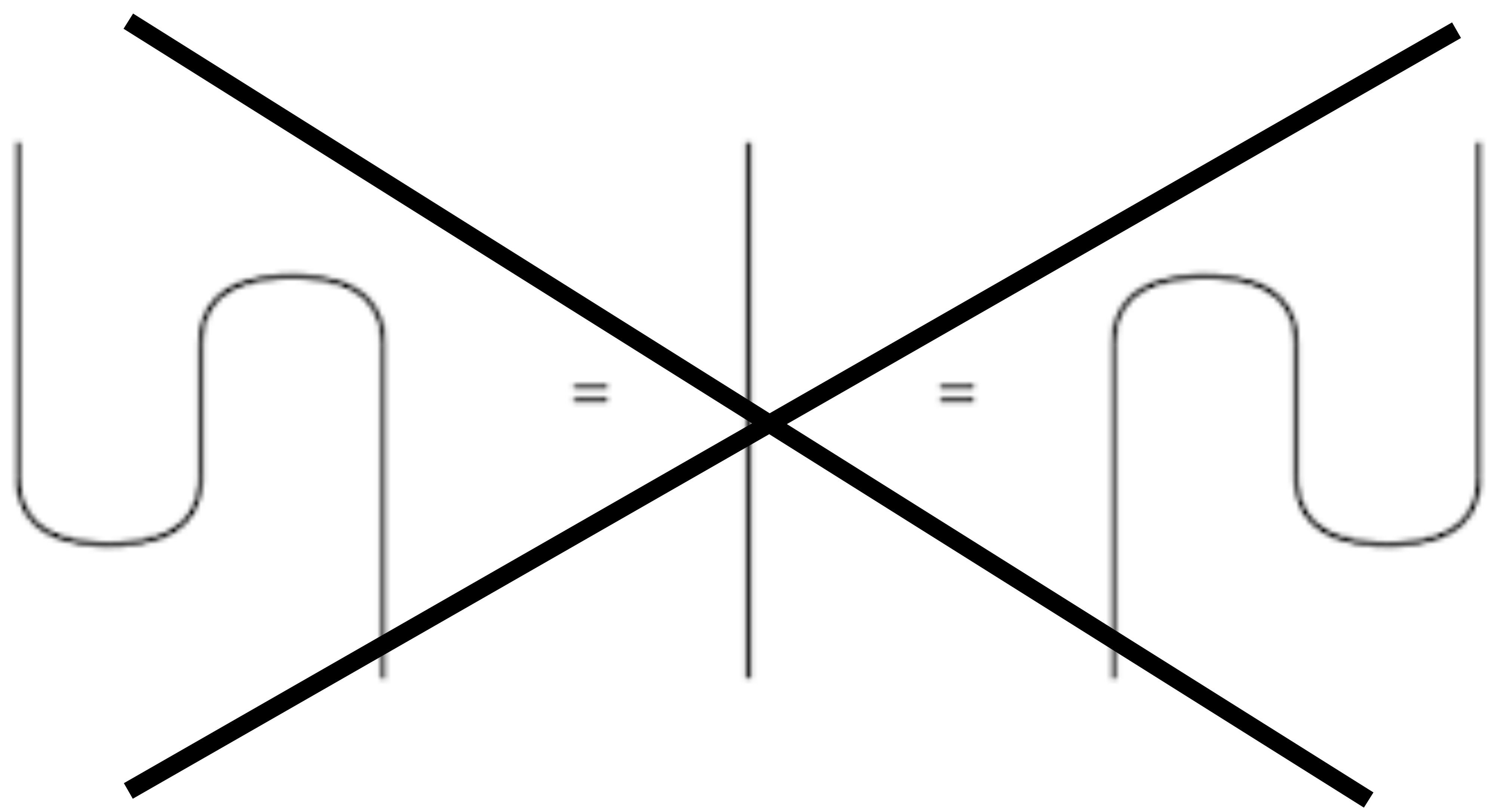


Why?

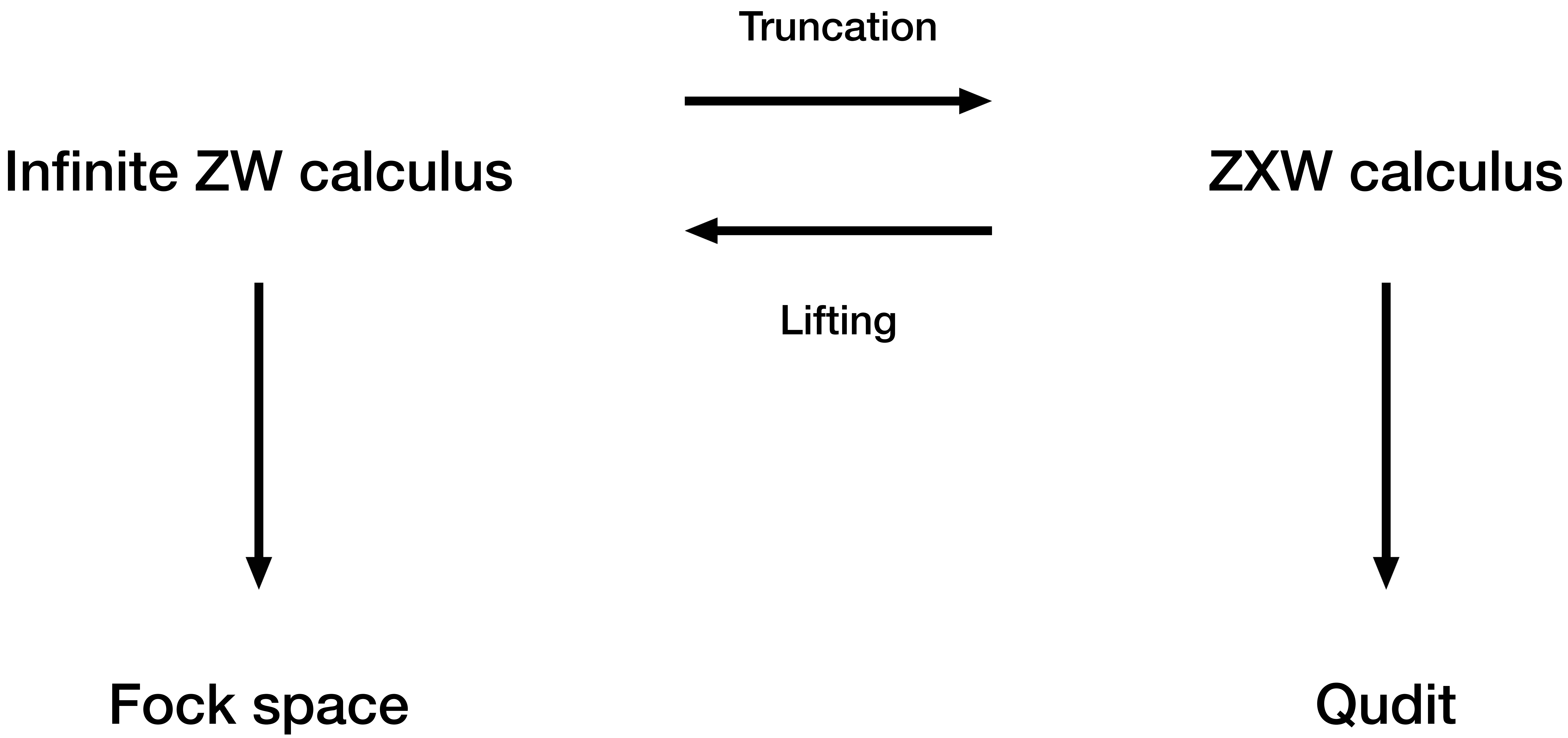
- Underlying structure of physical processes
- Formally verified software
- Optimisation and simulation by rewriting

Problem: infinite-dimensional vector spaces

- Photons live in infinite dimensional Fock space
- Natural equations that hold in finite dimensions fail in infinite dimensions.



Main results



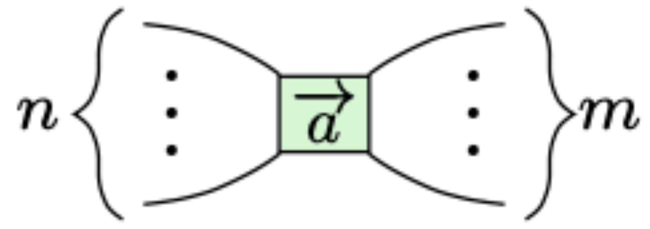
Outline

- ZXW calculus for qudits
- Fock space semantics
- Infinite ZW calculus
- Lifting theorem
- Quantum optical Hamiltonians

ZXW calculus

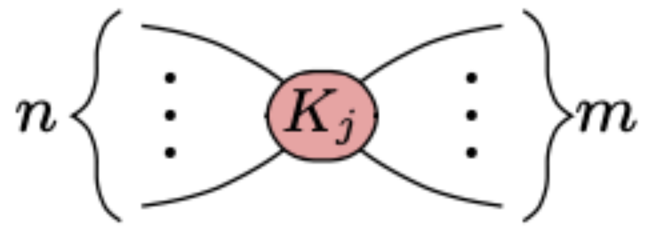
Generators

- The *Z spider*,



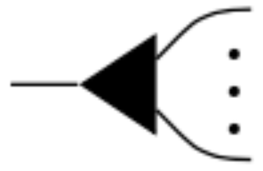
$$\xmapsto{[\![\cdot]\!]_d} \quad \sum_{j=0}^{d-1} a_j \, |j\rangle^{\otimes m} \langle j|^{\otimes n}, \quad \text{where } a_0 = 1.$$

- The *X spider*, with parameter j which can be taken modulo d,



$$\xmapsto{[\![\cdot]\!]_d} \quad \sum_{\substack{0 \le i_1, \cdots, i_m, j_1, \cdots, j_n \le d-1 \\ i_1 + \cdots + i_m + j \equiv j_1 + \cdots + j_n \pmod{d}}} |i_1, \cdots, i_m\rangle \langle j_1, \cdots, j_n|,$$

- The *W node*,

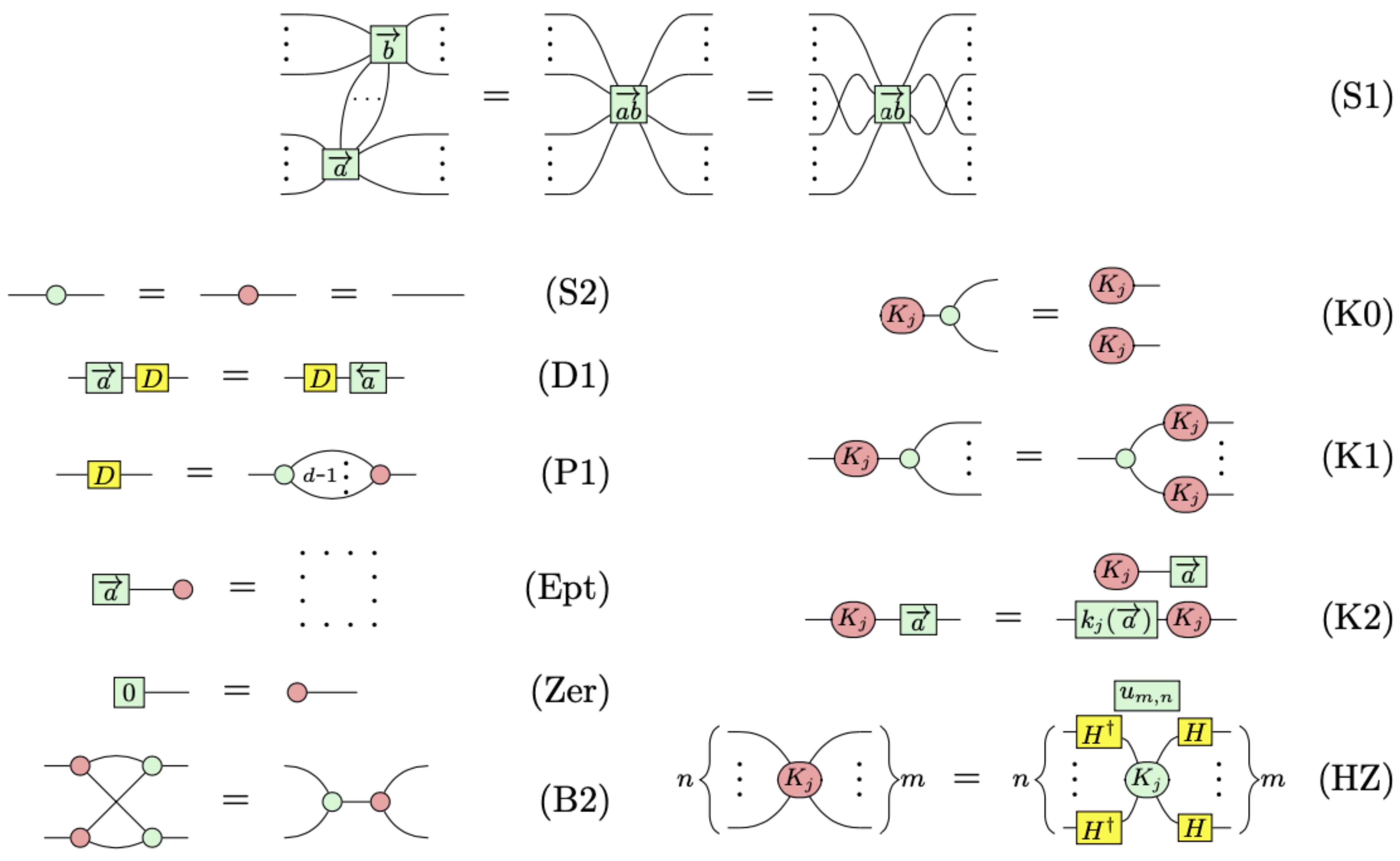


$$\xmapsto{[\![\cdot]\!]_d} \quad |0 \cdots 0\rangle \langle 0| + \sum_{i=1}^{d-1} (|i0 \cdots 00\rangle + \cdots + |00 \cdots 0i\rangle) \langle i|$$

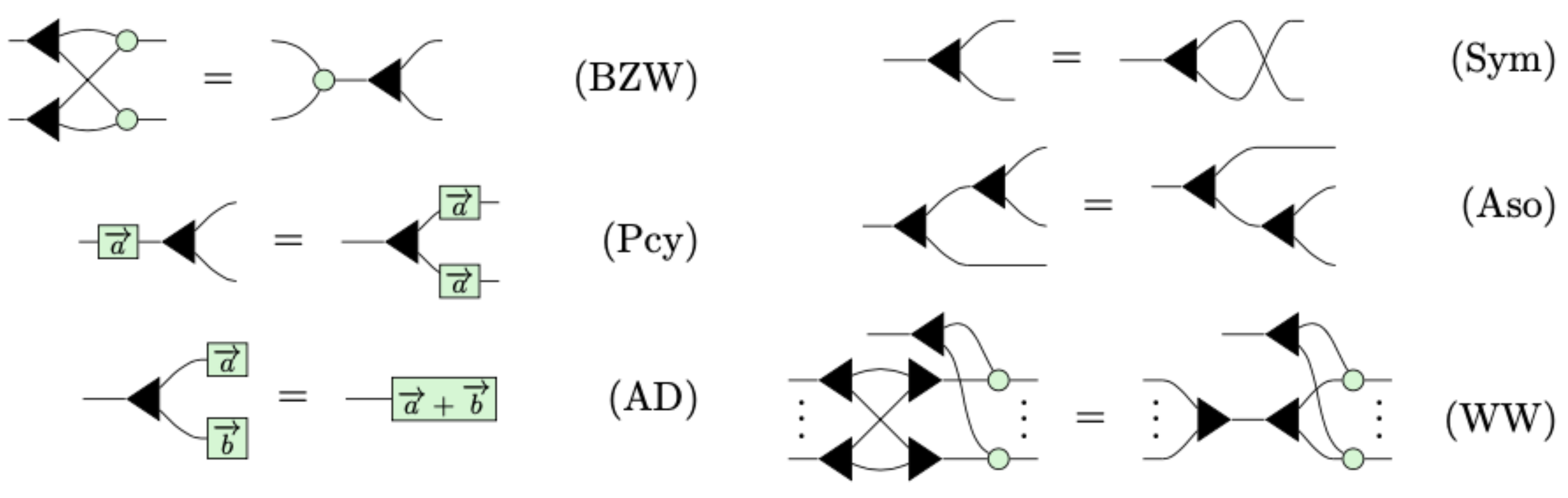
ZXW calculus

Complete axiomatisation

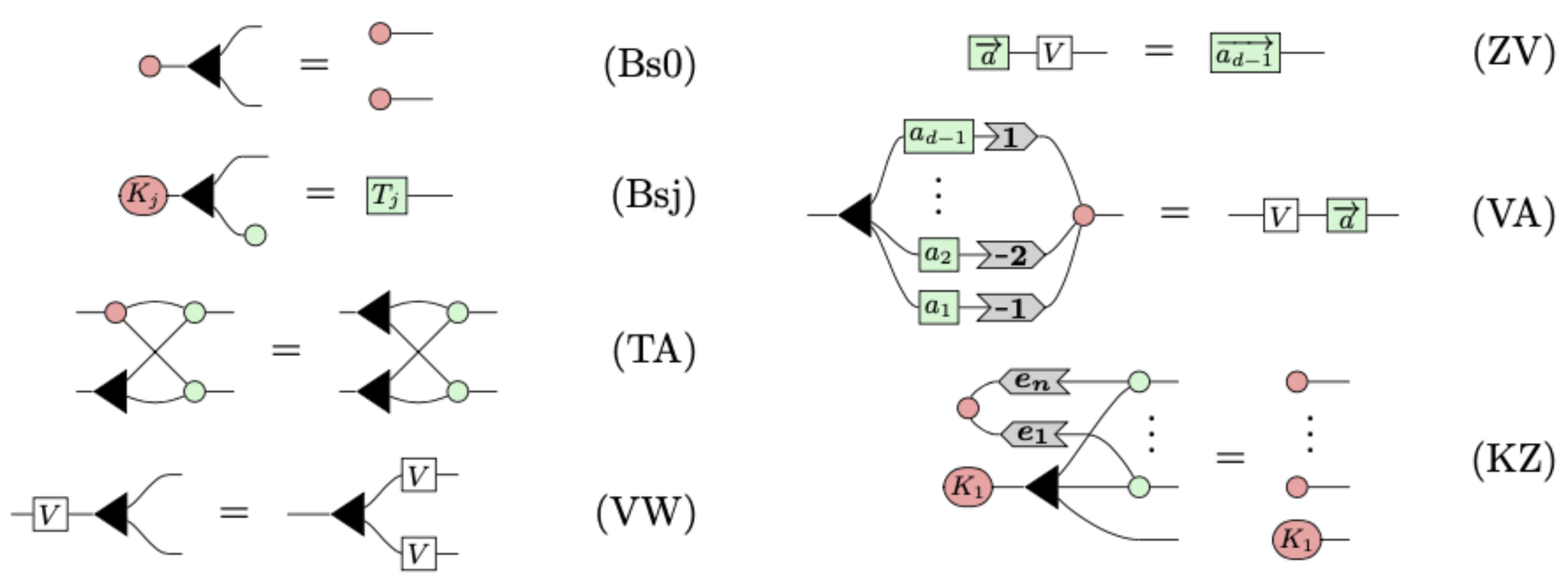
Qudit ZX-part of the rules



Qudit ZW-part of the rules

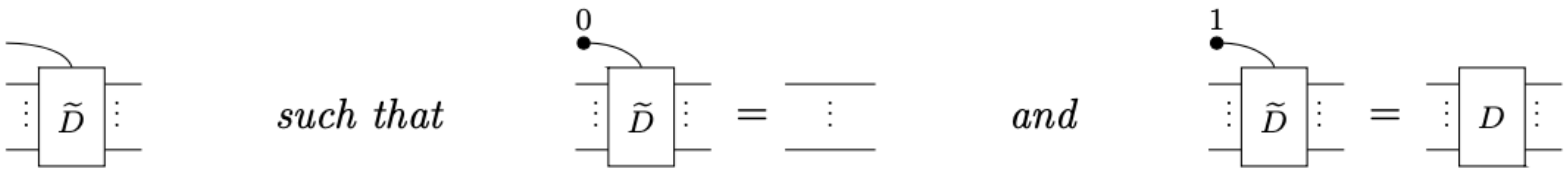


Qudit ZXW-part of the rules

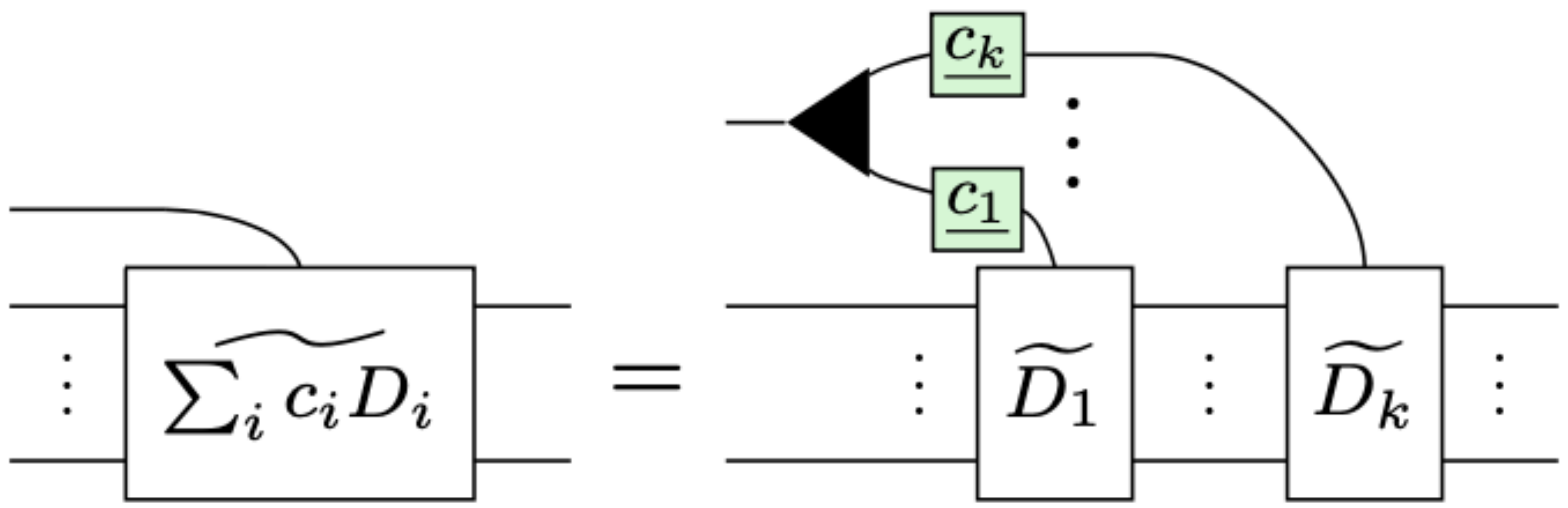


Sums of diagrams in ZXW

Controlled diagram



Sum of diagrams



Infinite ZW calculus

Fock space semantics

State space for a single bosonic mode: $[\![1]\!] = \bigoplus_{n=0}^{\infty} \mathbb{C}$

For multiple modes: $$[\![m]\!] = \bigoplus_{n=0}^{\infty} (\mathbb{C}^m)^{\tilde{\otimes} n} \simeq [\![1]\!]^{\otimes m}$$

Infinite direct sum : valid states are sequences with finitely many non-zero terms

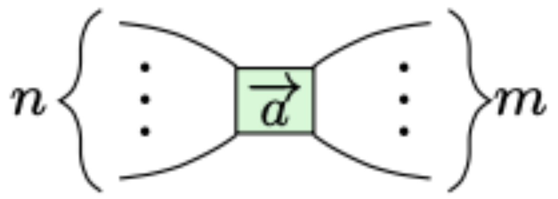
$\mathbf{Vect}_{\mathbb{N}}$

- Objects: finite tensor products of [[1]]
- Arrows: linear maps between them

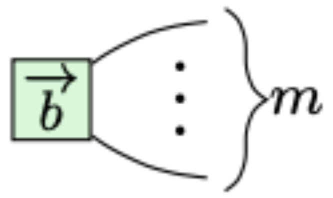
Infinite ZW calculus

Generators

Z nodes:

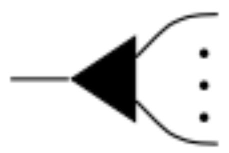


$$\overset{[\![\cdot]\!]}{\longmapsto} \quad \sum_{i=0}^{\infty} a_i \left|i\right\rangle^{\otimes m} \left\langle i\right|^{\otimes n}, \quad \text{where } n > 0, \overrightarrow{a} = (a_1, \cdots, a_k, \cdots),\ a_0 := 1.$$



$$\overset{[\![\cdot]\!]}{\longmapsto} \quad \sum_{i=0}^{N} a_i \left|i\right\rangle^{\otimes m}, \quad \text{where } n > 0, \overrightarrow{a} = (a_1, \cdots, a_N, 0, \cdots),\ a_0 := 1.$$

W nodes:

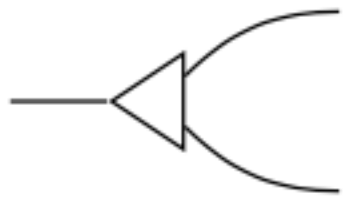


$$\overset{[\![\cdot]\!]}{\longmapsto} \quad \left|0\cdots 0\right\rangle \left\langle 0\right| + \sum_{i=1}^{\infty} (\left|i0\cdots 0\right\rangle + \cdots + \left|0\cdots 0i\right\rangle) \left\langle i\right|$$

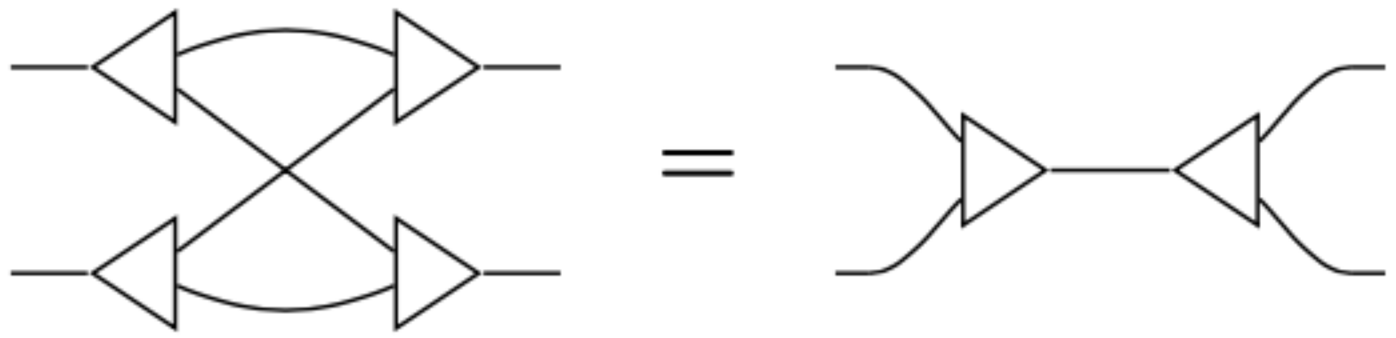
Infinite ZW calculus

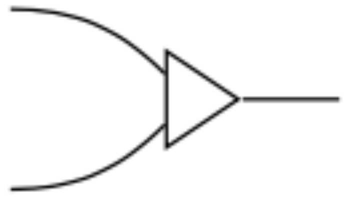
Generators

Bosonic nodes:



$$\xmapsto{[\![\cdot]\!]} \quad |n\rangle \mapsto \sum_{k=0}^{n} \binom{n}{k}^{\frac{1}{2}} |k\rangle\,|n-k\rangle$$





$$\xmapsto{[\![\cdot]\!]} \quad |n, m\rangle \mapsto \binom{n+m}{n}^{\frac{1}{2}} |n+m\rangle$$

n-Photon states and effects:

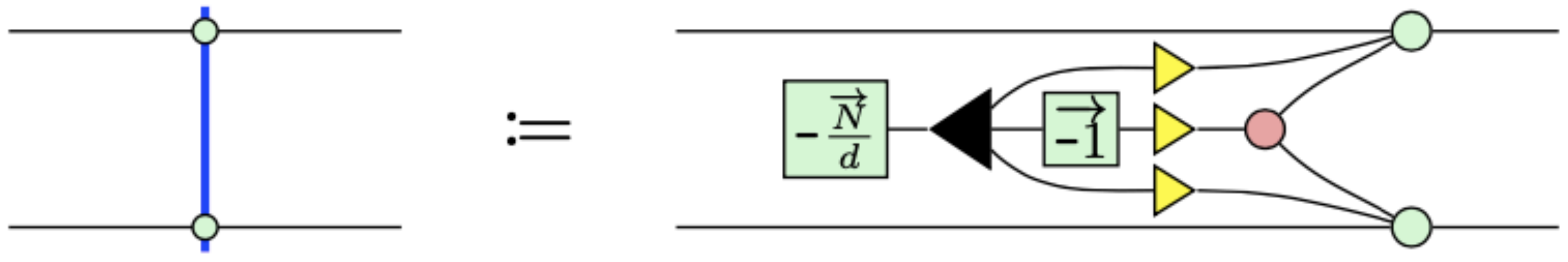
$$\xmapsto{[\![\cdot]\!]} \quad |n\rangle$$

$$\xmapsto{[\![\cdot]\!]} \quad \langle n|$$

Truncation

$$\mathcal{T}_d : \mathbf{ZW}_\infty \to \mathbf{ZXW_d}$$

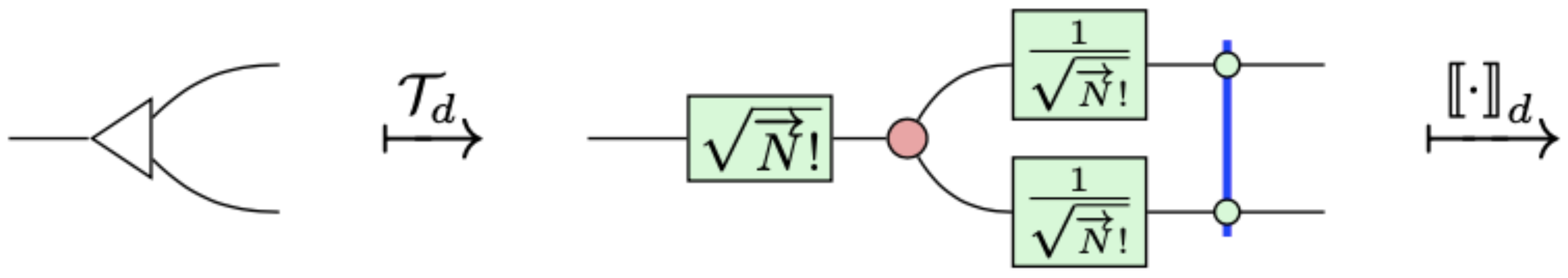
Projector on the d-particle sector:



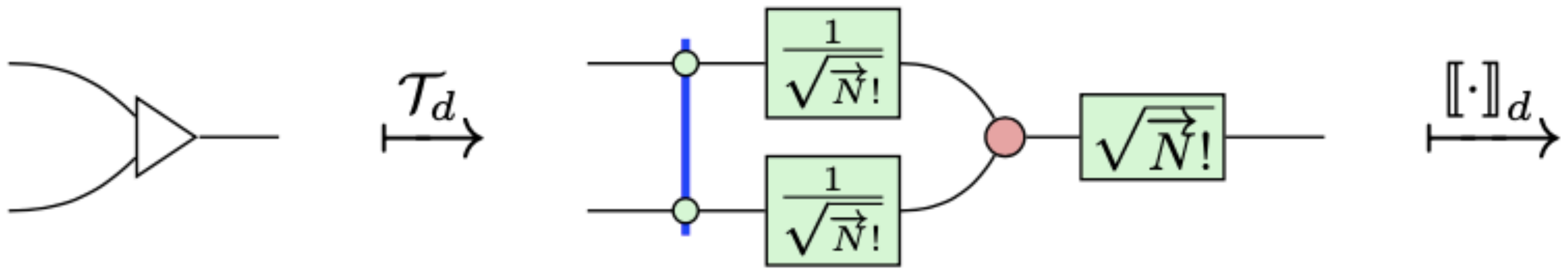
$$\xmapsto{[\![\cdot]\!]_d} \sum_{\substack{a_1,a_2=0 \\ a_1+a_2<d}}^{d-1} |a_1, a_2\rangle \langle a_1, a_2|$$

Truncation

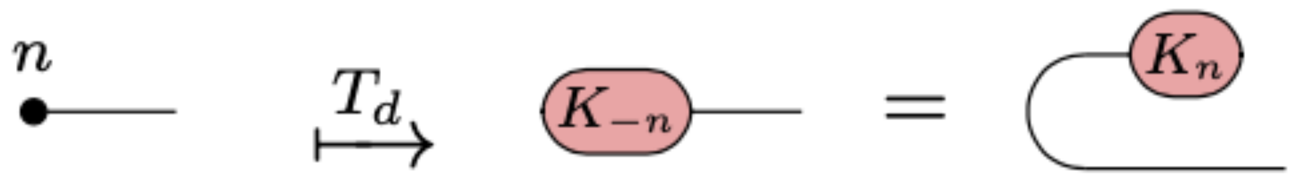
$$\mathcal{T}_d : \mathbf{ZW}_\infty \to \mathbf{ZXW_d}$$

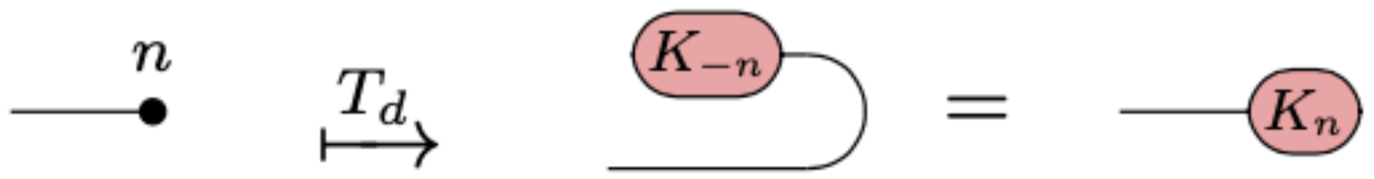


$$\sum_{n=0}^{d-1} \sum_{k=0}^{n} \binom{n}{k}^{\frac{1}{2}} |k, n-k\rangle \langle n|$$



$$\sum_{n,m=0 n+m<d}^{d-1} \binom{n+m}{n}^{\frac{1}{2}} |n+m\rangle \langle n,m|$$

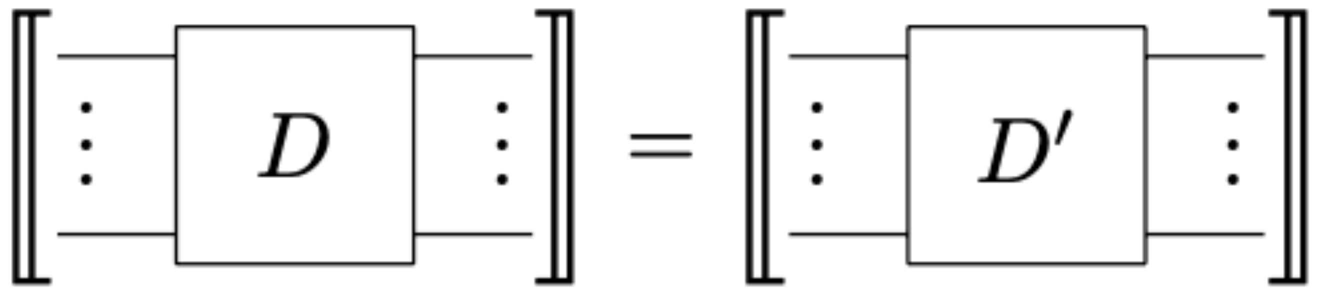




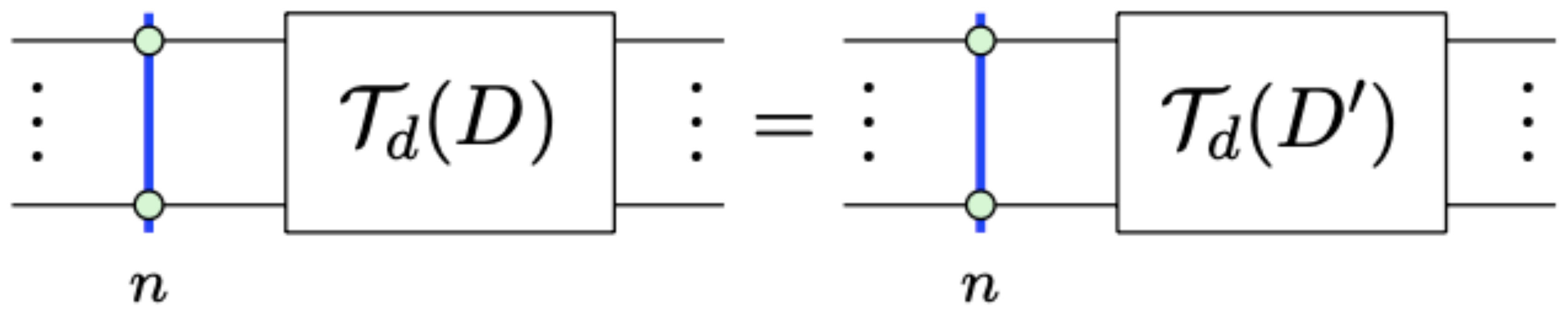
Lifting theorem

Theorem 4.2 (Lifting)**.** *For any* $D, D' : m \to m' \in \mathbf{ZW}_\infty$ *the following are equivalent:*

1. *In* $\mathbf{Vect}_{\mathbb{N}}$*:*



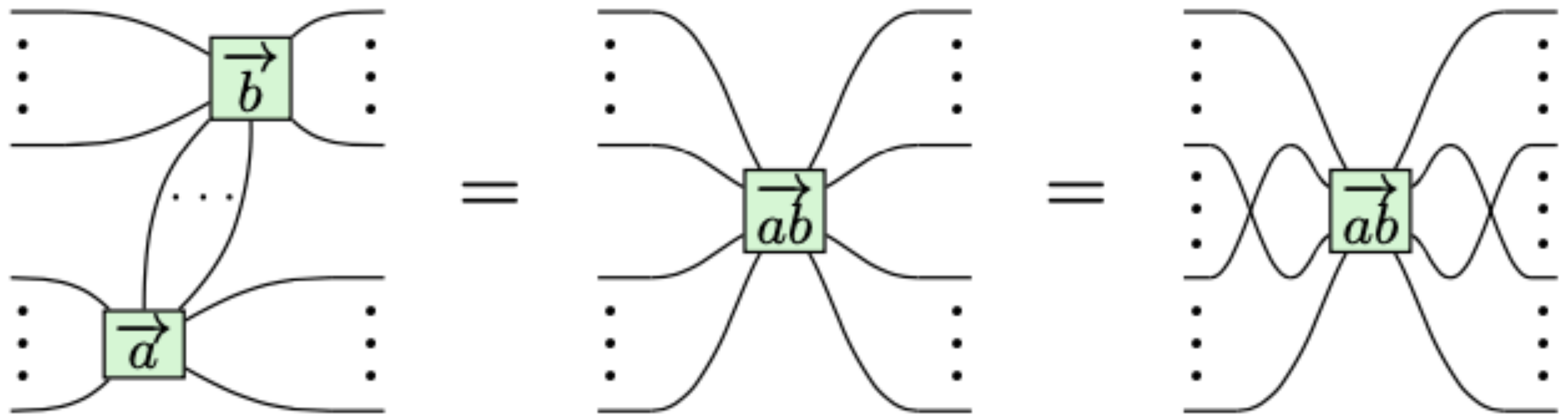
2. *For any* $n \in \mathbb{N}$ *there is a dimension* d^* *such that, for all* $d > d^*$*, in* $\mathbf{ZXW_d}$*:*



Infinite ZW calculus

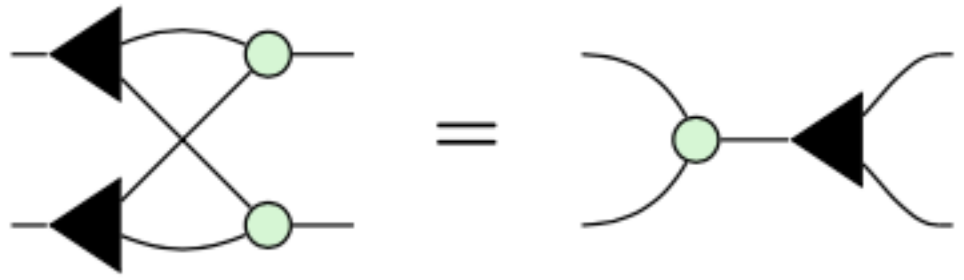
Axioms

Non-unital Frobenius algebra

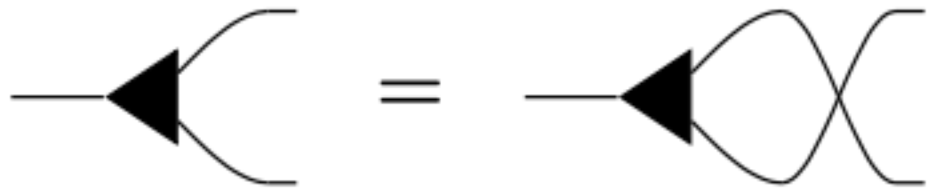


($S1_\infty$)

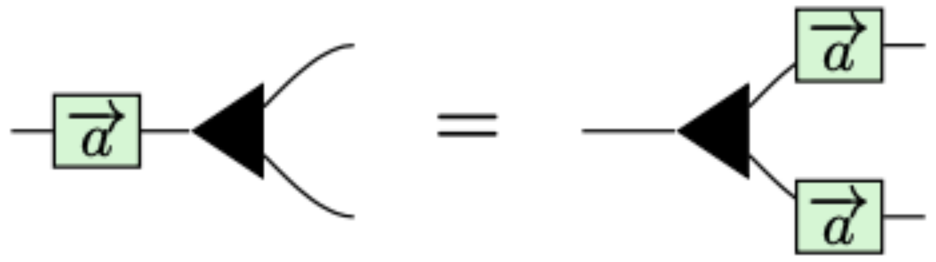
Rules generalised from qudit ZW



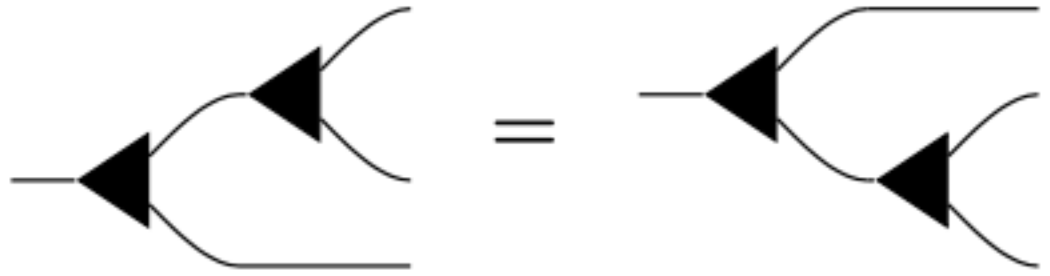
(BZW_∞)



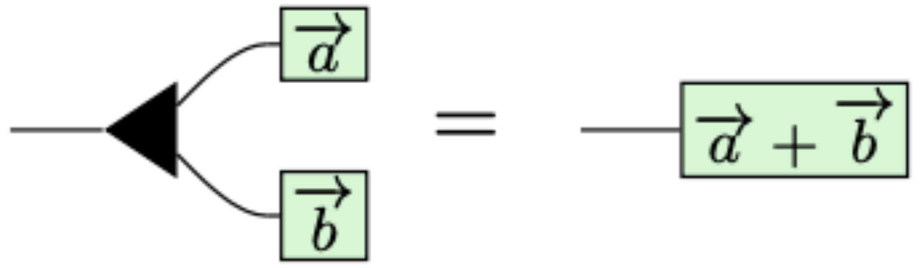
(Sym_∞)



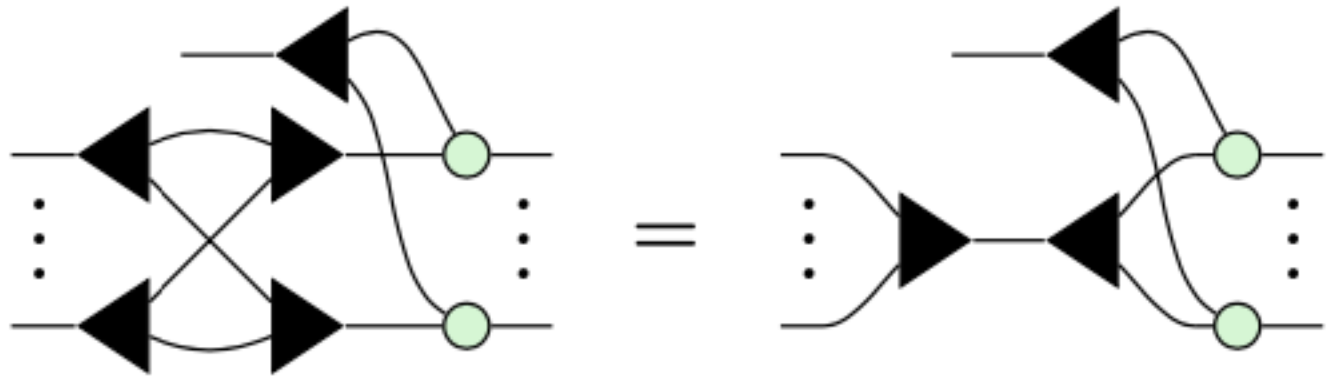
(Pcy_∞)



(Aso_∞)



(AD_∞)

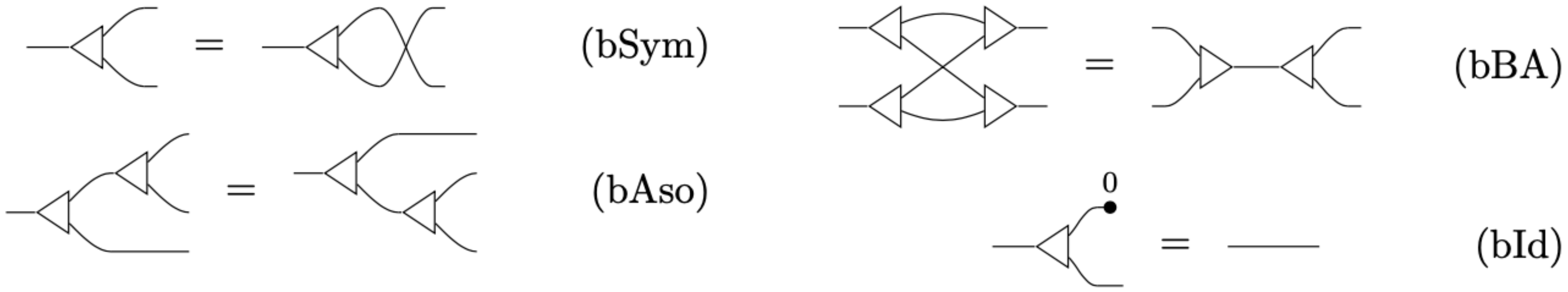


(WW_∞)

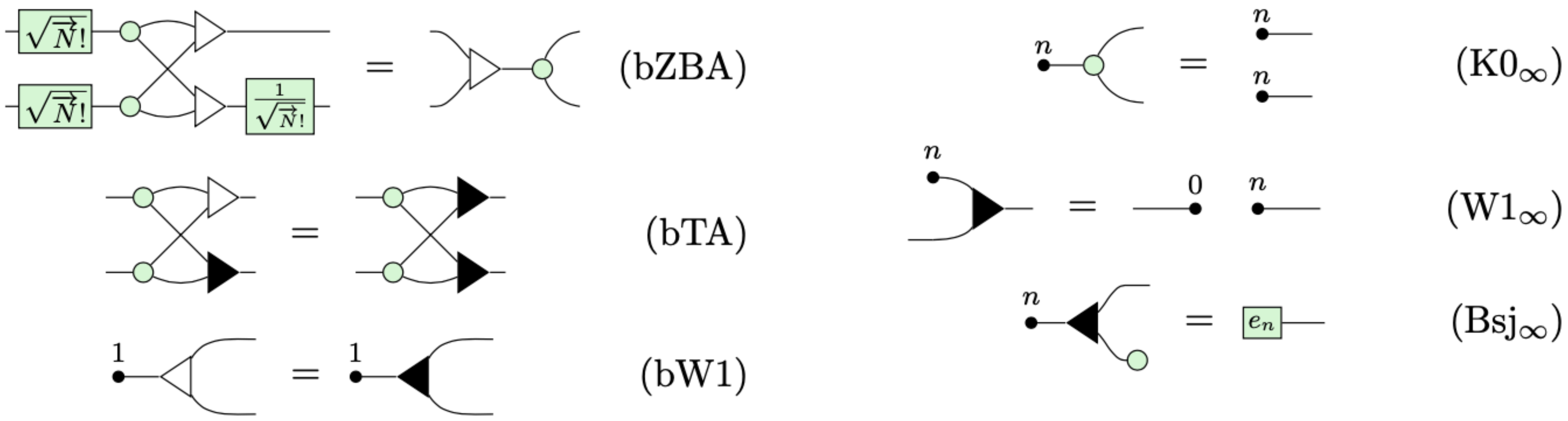
Infinite ZW calculus

Axioms

Rules from QPath



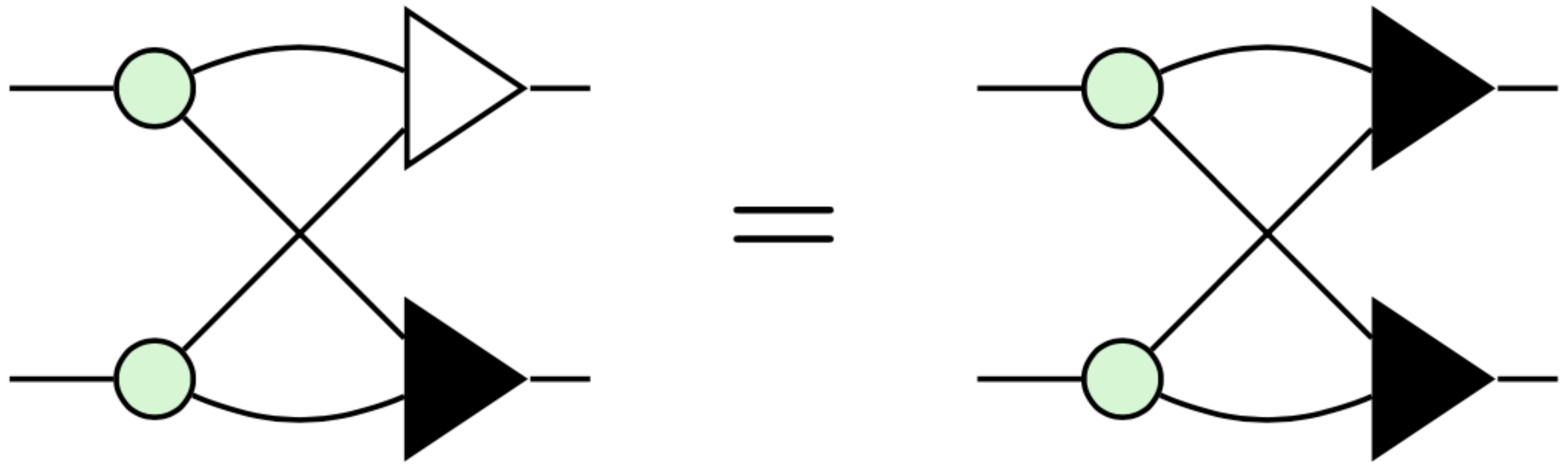
Interaction rules



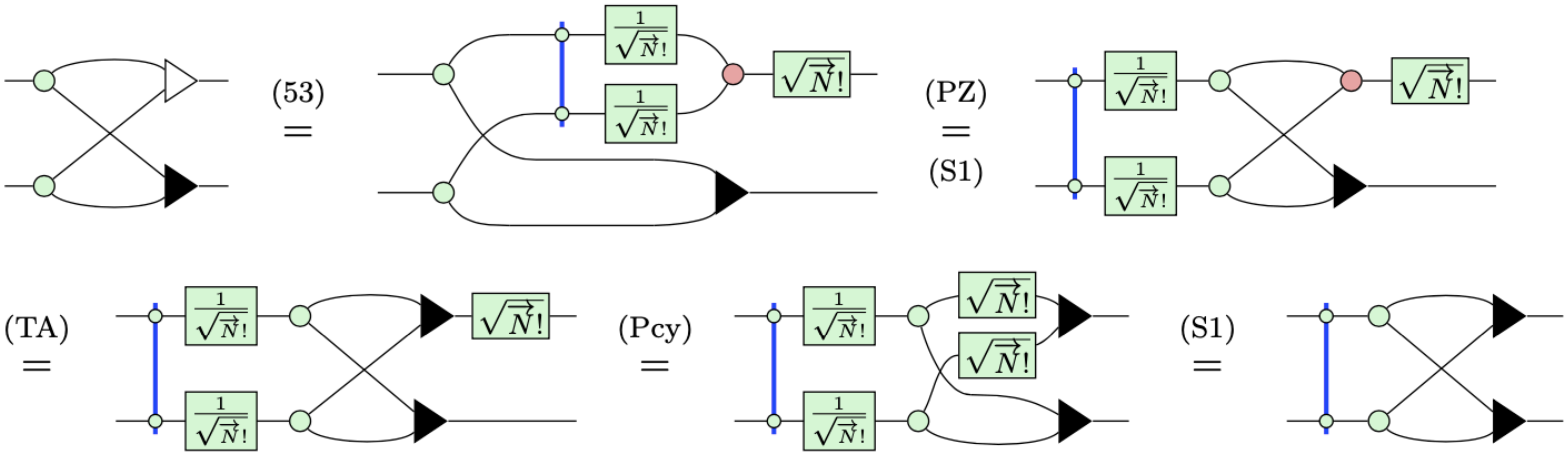
where $e_n = (\underbrace{0, \dots, 1}_{n}, 0, \dots)$

Soundness

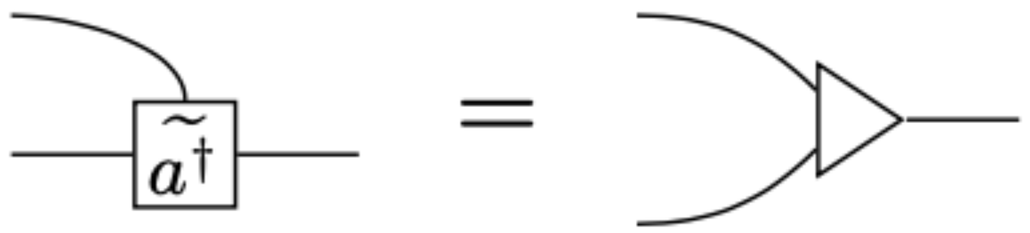
Proposition:

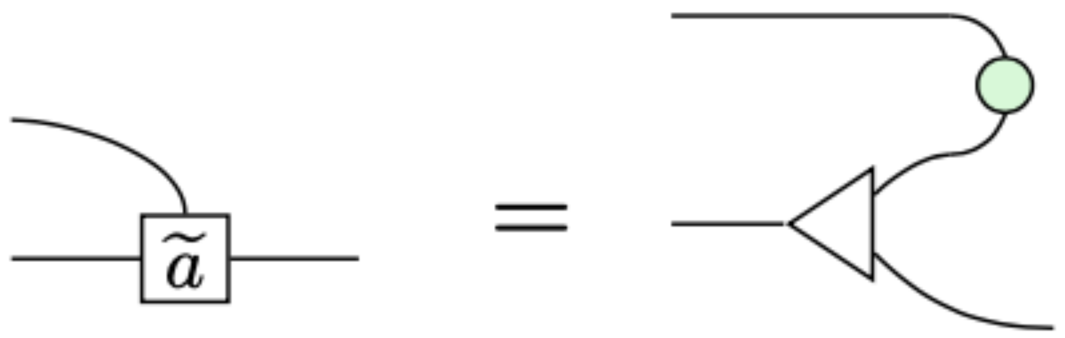


Proof:

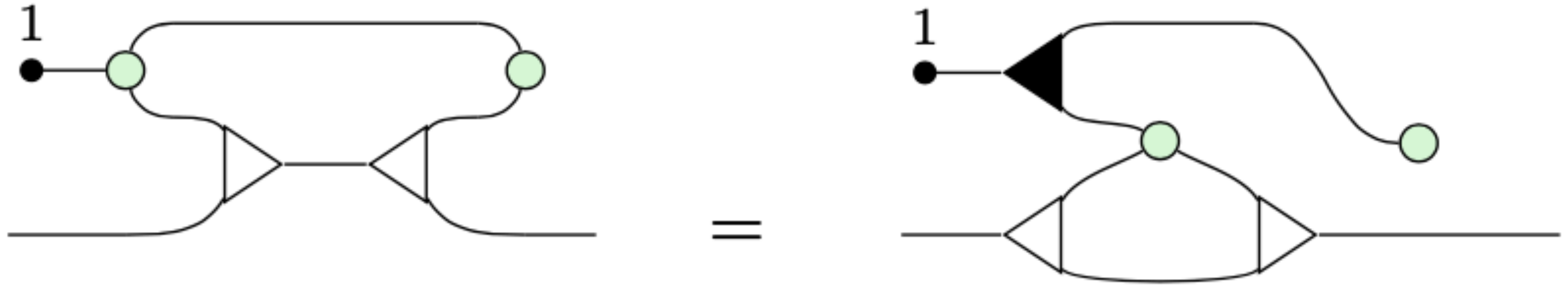


Applications
Bosonic creation and annihilation





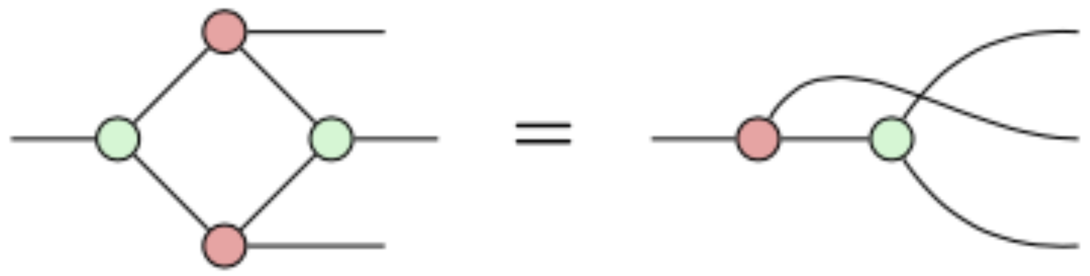
Proposition 5.4. *In* $\mathbf{ZW}_\infty$, $aa^\dagger = a^\dagger a + id$, *that is:*



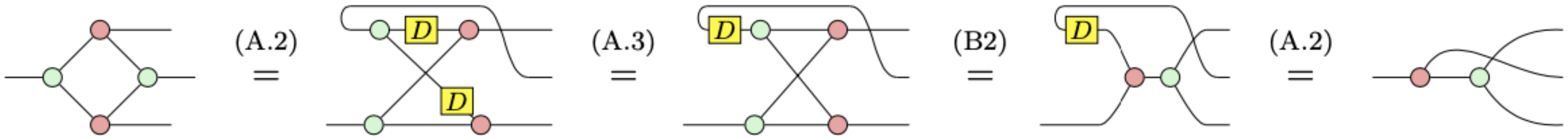
Applications

Bosonic creation and annihilation

Lemma A.4. *In* $\mathbf{ZXW_d}$, *for any d, we have:*



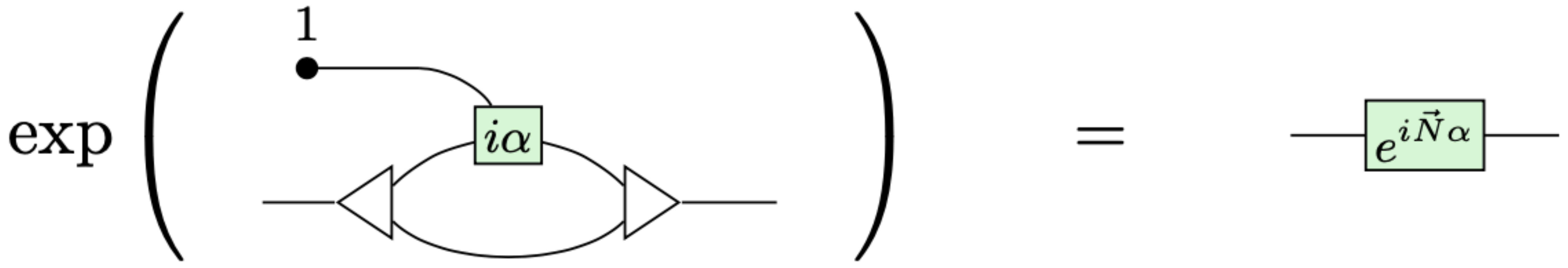
Proof.



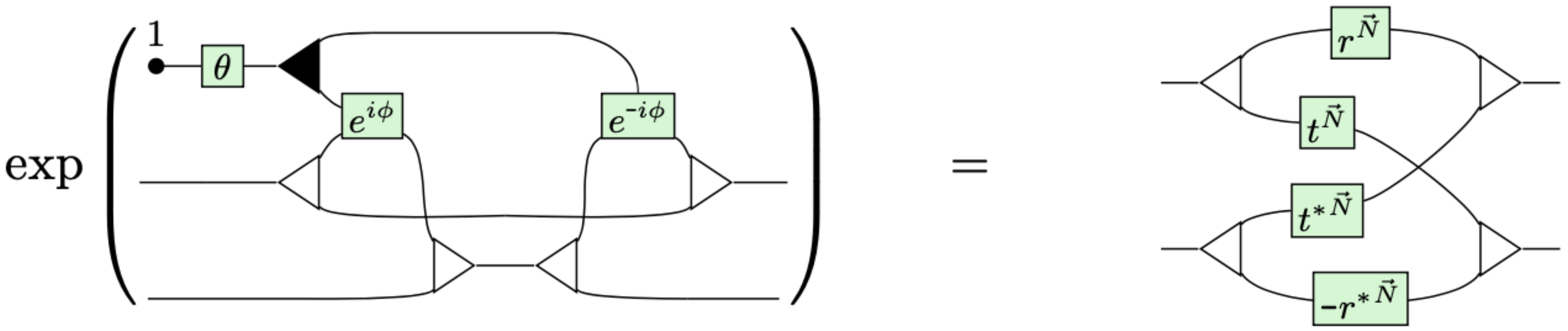
□

Applications
Linear optics

Phase shift: $$H_P = \alpha \hat{n}_1 = \alpha a_1^\dagger a_1$$



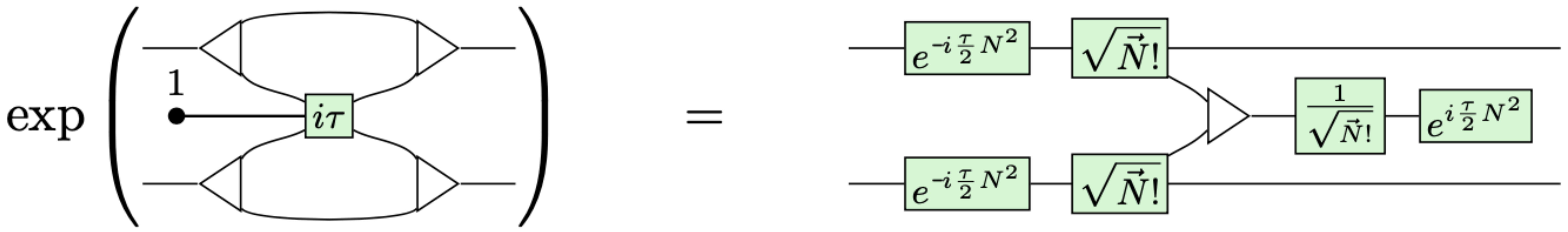
Beam splitter: $$H_{BS} = \theta \left(e^{i\phi} a_1 a_2^\dagger + e^{-i\phi} a_1^\dagger a_2 \right)$$



Applications

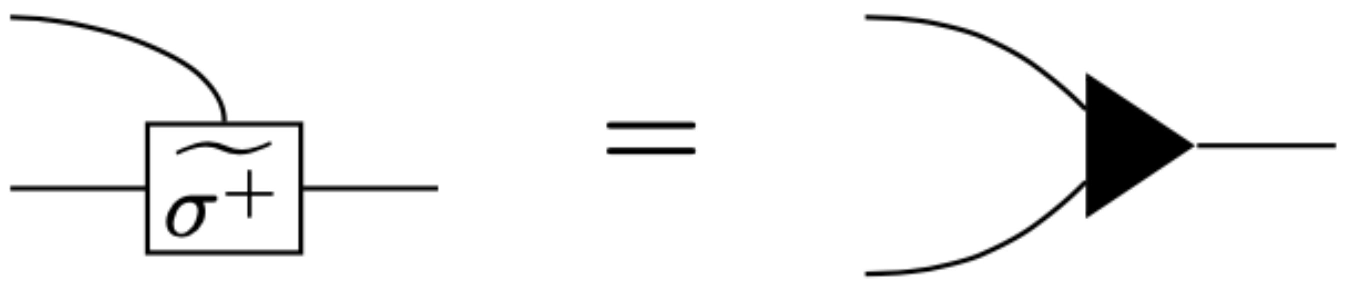
cross-Kerr media as phase gadgets

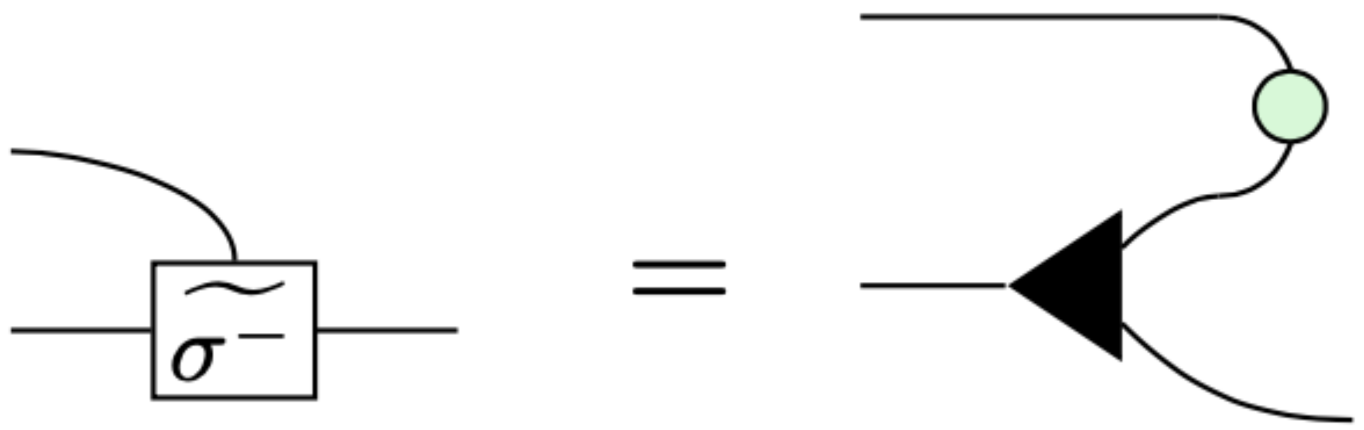
$$H_{CK} = \tau \hat{n}_1 \hat{n}_2 = \tau a_1^\dagger a_1 a_2^\dagger a_2$$



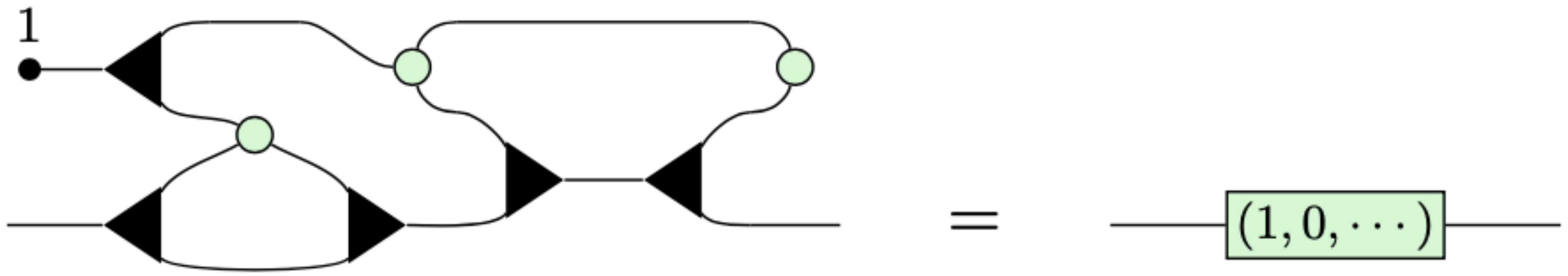
Applications

Fermionic creation and annihilation





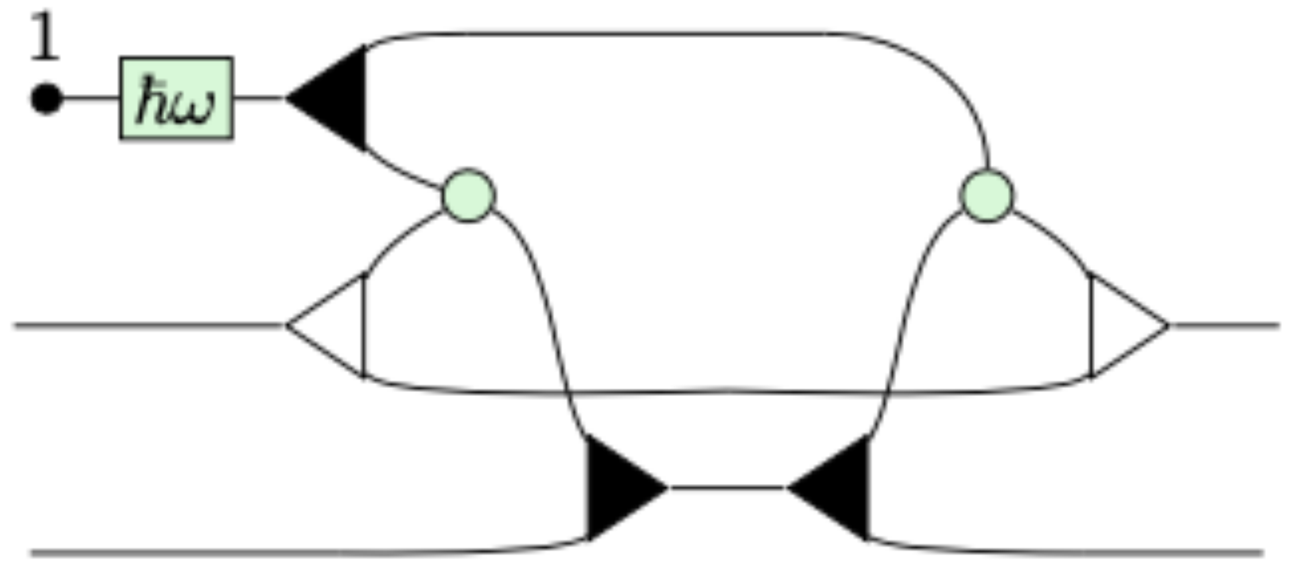
Proposition 5.9. *In* $\mathbf{ZW}_\infty$, $\sigma^-\sigma^+ + \sigma^+\sigma^- = id$, *that is:*

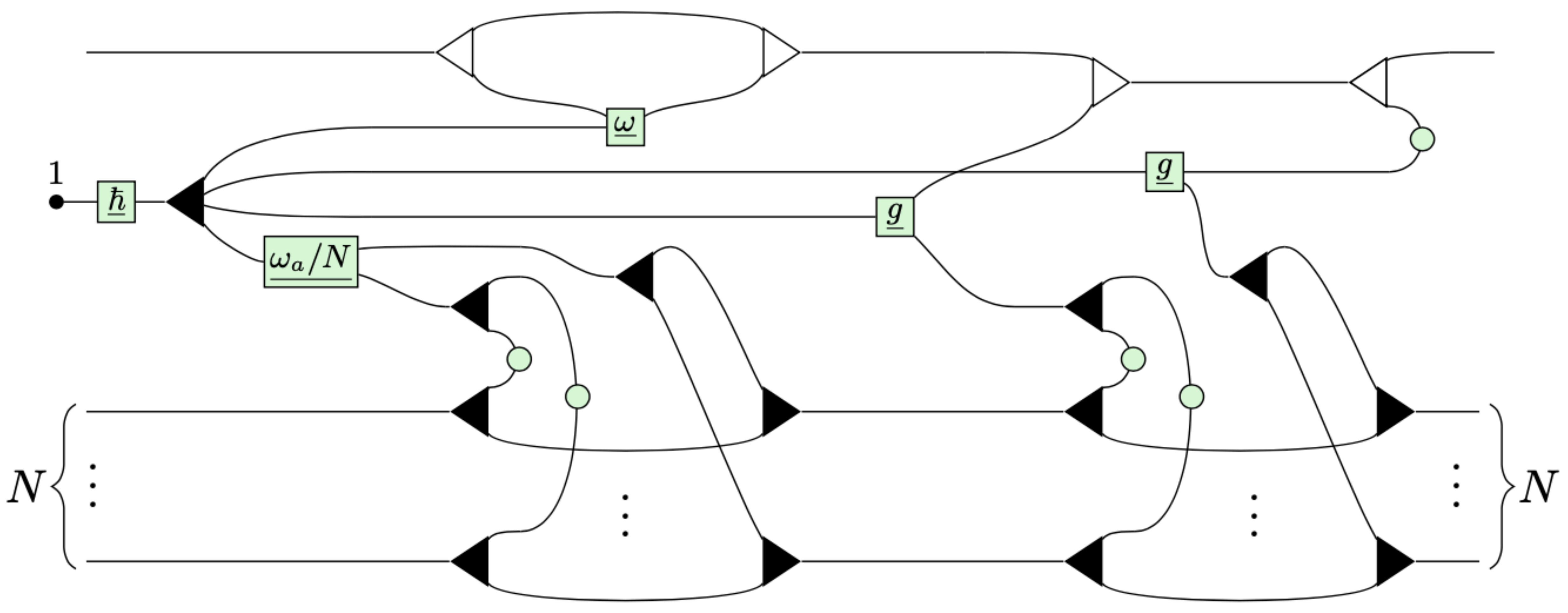


Applications

Jaynes-Cummings Hamiltonian

$$H_{JC} = \hbar\omega \left(a_1 \sigma_2^+ + a_1^\dagger \sigma_2^- \right)$$





Future work

- Hamiltonian simplification
- Tensor network simulation
- Measurements, classical control and lossy channels (CPM construction)
- Coherent states and continuous variable photonics (Gaussian states, squeezed states, cat qubits, …)