Self-duality and Jordan structure of quantum theory follow from homogeneity and pure transitivity

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Homogeneity + pure transitivity $\implies$ *self-duality.*

From this follows:

Homogeneity + pure transitivity + local tomography uniquely defines quantum theory

What is self-duality?

In a **Generalised probabilistic theory** (GPT) we describe a system by a

- convex **state space** Ω,
- convex **effect space** E,
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In quantum theory:

- $\Omega \subseteq M_n(\mathbb{C})_{\text{sa}}$ are density matrices,
- $E \subseteq M_n(\mathbb{C})_{\text{sa}}$ are positive sub-unital matrices,
- $(\cdot,\cdot)$ given by inner product $\langle A, B\rangle := \text{tr}\,(AB)$.

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Something peculiar: Ω and E belong to **same** space $M_n(\mathbb{C})_{\text{sa}}$,
and are related by inner product.
This is **self-duality**.

Self-duality

Definition (informal)

A system is **self-dual** when (unnormalized) states can be identified with the effects by a probability-determining inner product.

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Self-duality is 'rare' amongst GPTs:

Koecher-Vinberg theorem

self-duality + *homogeneity* = Jordan algebra = 'almost' quantum.

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A: General inner products don't map valid states to valid effects.

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- Desired inner product should hence at least preserve positivity.

Self-dual inner product

Definition

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Equivalently: view $\langle \cdot, \cdot \rangle$ as $\Phi : V \to V^*$ by $\Phi(v)(w) = \langle v, w \rangle$.
Then $\langle \cdot, \cdot \rangle$ is self-dual iff Φ is an **order isomorphism**:

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Note

The existence of just an order iso $\Phi : V \to V^*$ is known as **weak** self-duality. Weak SD is necessary for state-teleportation protocols in GPTs (Barnum *et al.* 2012).

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- Φ is certainly positive. Can also easily construct a positive inverse.
- Hence Φ is an order iso.
- And we see that $\Phi(\rho) = \sigma$.

So quantum systems are homogeneous.

Homogeneity operationally

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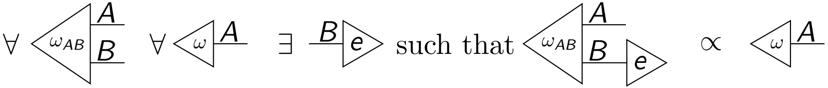
Informally, we say a system B **universally steers** A, if for every bipartite state ω_{AB} we can induce any* state on A by observing the right effect on B.



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$\Rightarrow$ EJAs are 'almost-quantum' systems.

So what now?

- Koecher-Vinberg theorem is very powerful.
- Homogeneity has operational interpretation (steering).
- Self-duality does not.
- Can we replace it with some other nicer/operational property?

Pure transitivity

Definition

In a GPT system, a **pure state** is a convex extremal element of Ω:

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- and we consider reversible transformations as the 'real' dynamics,
- then failure of pure transitivity would mean two states of a system are not transformable into each other.
- But then isn't our definition of system is wrong?

Comparing homogeneity and pure transitivity

Recall $\Omega \subseteq V_+ \subseteq V$.

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for all pure $\omega_1, \omega_2 \in \Omega$ there exists a normalised order iso Φ such that $\Phi(\omega_1) = \omega_2$.

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for all interior $v_1, v_2 \in V_+$ there exists an order iso Φ such that $\Phi(v_1) = v_2$.

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Note: order iso's are rescaled **probabilistically reversible** transformations: $\Phi \circ \Phi^\sharp = p\,\mathrm{id}$

Our results

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Corollary (via Koecher-Vinberg theorem)

Such a vector space is then order-isomorphic to a Euclidean Jordan algebra.

Some more corollaries

Definition

We say Ω satisfies **continuous** pure transitivity when for all pure $\omega_1, \omega_2 \in \Omega$ there is a family Φ_t of reversible transformations for $t \in [0, 1]$ such that $t \mapsto \Phi_t(v_1)$ is a continuous path from v_1 to v_2.

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Corollary

The state space of a system that satisfies continuous pure transitivity and universal self-steering is order-isomorphic to a Euclidean Jordan algebra.

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Theorem

Let A be a system in a GPT where composites are locally tomographic and every state space is homogeneous and satisfies continuous pure transitivity. Then $V_A \cong M_n(\mathbb{C})_{sa}$.

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- But $\Phi(V^c) = V^c$, so $\omega \in V^c$.
- Hence V and V^c have the same pure states.

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Theorem

Let V be a homogeneous ordered vector space that satisfies pure transitivity. Then V is self-dual.

- Vinberg (1963) showed that each homogeneous V has a non-zero subspace V^c, such that
- $V_+^c := V^c \cap V_+$ is homogeneous and **self-dual**.
- It turns out V^c is invariant under normalised order iso's of V.
- V^c has at least one pure state ω_c that is also a pure state of V.
- Now let ω in V be pure. With pure transitivity we find a normalised order iso Φ such that $\Phi(\omega_c) = \omega$.
- But $\Phi(V^c) = V^c$, so $\omega \in V^c$.
- Hence V and V^c have the same pure states.
- Hence $V^c = V$.

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- Self-duality follows from homogeneity and pure transitivity.
- Homogeneity and pure transitivity have an operational interpretation, so this gives an operational variant of the Koecher-Vinberg theorem.
- Also requiring local tomography uniquely pinpoints quantum theory.
- Could've instead assumed a 'dynamical correspondence': a mapping from reversible transformations to observables.
- This then hence gives a reconstruction purely in terms of the symmetries of the pure and mixed states.

Thank you for your attention!

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Self-duality and Jordan structure of quantum theory follow from homogeneity and pure transitivity