Moore-Penrose Dagger Categories

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Mt.Fuji Adventure

Start of hike.

Top of Mt.Fuji.
(Fun fact: this was on my bday)

End of hike.

Moore-Penrose Inverse of Complex Matrices

- For a $n \times m$ $\mathbb{C}$-matrix A, its **Moore-Penrose inverse** is an $m \times n$ $\mathbb{C}$-matrix A° satisfying:

$$AA^\circ A = A \qquad A^\circ AA^\circ = A^\circ \qquad (AA^\circ)^\dagger = AA^\circ \qquad (A^\circ A)^\dagger = A^\circ A$$

where $\dagger$ is conjugate transpose.

Theorem

For any $\mathbb{C}$-matrix, its Moore-Penrose inverse exists and is unique!

- One way to compute it is using singular value decomposition (SVD):

$$A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}_{n\times m} V^\dagger \qquad \text{where } D \text{ is the diagonal } k \times k \text{ matrix } D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_k \end{bmatrix}$$

where U and V are unitary matrices, and d_i are the non-zero singular values of A, so $d_i \in \mathbb{R}$ with $d_i > 0$. Then the M-P inverse of A is the $m \times n$ matrix A° defined as follows:

$$A^\circ = V \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{m\times n} U^\dagger \quad \text{where } D^{-1} \text{ is the diagonal } k \times k \text{ matrix } D^{-1} = \begin{bmatrix} \frac{1}{d_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{d_k} \end{bmatrix}$$

Since M-P inverses are unique, the construction does not depend on the choice of SVD.

History of Moore-Penrose Inverse is fascinating!

- Named after:

E.H. Moore

Roger Penrose

E.H. Moore **On the reciprocal of the general algebraic matrix**. (1920)

R. Penrose **A generalized inverse for matrices**. (1955)

R. Rado **Note on generalized inverses of matrices**. (1956)

- For a full history, references, and interesting applications, highly recommend:

O. M. Baksalary & G. Trenkler **The Moore–Penrose inverse: a hundred years on a frontline of physics research**. (2021)

Generalizing the Moore-Penrose Inverse

The Moore-Penrose inverse can be generalized to other contexts besides complex matrices:

- One may consider the Moore-Penrose inverse of a matrix over an involutive (semi)ring.

  While the Moore-Penrose inverse may not always exist, for certain involutive (semi)rings it is possible to precisely characterize which matrices have Moore-Penrose inverses.

  This is still an active field of research!

- One can also consider Moore-Penrose inverses in involutive semigroups, C^*-algebras, etc.

- It is also possible to define the notion of Moore-Penrose inverses for bounded linear operators between Hilbert spaces, and to characterize precisely which have a Moore-Penrose inverse.

Following in this direction, one can in fact define the notion of a Moore-Penrose inverse for maps in **dagger categories**.

Moore-Penrose Inverses for Dagger Categories

It is straightforward to define the notion of a Moore-Penrose for a map in a dagger category.

- The existence and computations of Moore-Penrose inverses for maps in general dagger categories was studied by Puystjens and Robinson in series of papers in the 1980s:

R. Puystjens

D. W. Robinson

- **The Moore-Penrose inverse of a Morphism with Factorization**. (1981)
- **The Moore-Penrose Inverse of a Morphism in an Additive Category**. (1984)
- **EP Morphisms**. (1985)
- **Generalized Inverses of Morphisms with Kernels**. (1987)
- **Symmetric Morphisms and the Existence of Moore-Penrose Inverses**. (1990)

Since their work, there does not appear to have been any further development of Moore-Penrose inverses in dagger categories... But lots of have been done with dagger categories!

Today's Story

Thanks to categorical quantum foundations, the theory of dagger categories itself has undergone significant development since the early 2000s.

- **TODAY'S STORY:** revisit and renew the study of Moore-Penrose inverses in dagger categories using the dagger category theory that have been developed since Puystjens and Robinson's work.

Moore-Penrose Inverses for Dagger Categories

A **dagger category** is a pair $(\mathbb{X}, \dagger)$ consisting of a category $\mathbb{X}$ equipped with a dagger $\dagger$, which a contravariant functor $(\_)^\dagger : \mathbb{X} \to \mathbb{X}$ which is the identity on objects and involutive. So for each map $f : A \to B$, there is a chosen map of dual type $f^\dagger : B \to A$, called its adjoint, such that [1]:

$$1_A^\dagger = 1_A \qquad (fg)^\dagger = g^\dagger f^\dagger \qquad (f^\dagger)^\dagger = f$$

Definition

In a dagger category $(\mathbb{X}, \dagger)$, a **Moore-Penrose inverse** (M-P inverse) of a map $f : A \to B$ is a map $f^\circ : B \to A$ such that the following equalities hold:

$$ff^\circ f = f \qquad f^\circ ff^\circ = f^\circ \qquad (ff^\circ)^\dagger = ff^\circ \qquad (f^\circ f)^\dagger = f^\circ f$$

A **Moore-Penrose dagger category** is a dagger category such that every map is M-P invertible.

Lemma

In a dagger category, M-P inverses are unique (if they exist).

[1] Composition is written diagramatically

Nice Moore-Penrose Identities

Lemma

In a dagger category $(\mathbb{X}, \dagger)$, if f has an M-P inverse f° then:

- *f° is also M-P invertible where $f^{\circ\circ} = f$;*
- *$f^\dagger$ is also M-P invertible where $f^{\dagger\circ} = f^{\circ\dagger}$*

Lemma

In a dagger category $(\mathbb{X}, \dagger)$:

- *Identity maps 1_A are M-P invertible where $1_A^\circ = 1_A$;*
- *If f is an isomorphism, then f is M-P invertible where $f^\circ = f^{-1}$;*
- *If f is a partial isometry ($ff^\dagger f = f$) then f is M-P invertible where $f^\circ = f^\dagger$;*
- *Isometries, coisometries, unitary maps, $\dagger$-idempotents are all M-P invertible.*

WARNING!: even if f and g have M-P inverses, fg might not have an M-P inverse and, even if it does, $(fg)^\circ$ is not necessarily equal to $g^\circ f^\circ$.

See our paper for some conditions for when $(fg)^\circ = g^\circ f^\circ$ holds.

Examples 1

Example

(MAT($\mathbb{C}$), $\dagger$) is a Moore-Penrose dagger category.

Example

(MAT($\mathbb{C}$), T) is NOT a Moore-Penrose dagger category. For example, the matrix $\begin{bmatrix} i & 1 \end{bmatrix}$ does not have a M-P inverse with respect to the transpose.

Example

(HILB, $\dagger$) is NOT a Moore-Penrose dagger category but we can precisely characterize which maps do have M-P inverses:

- A bounded linear operator $f : H_1 \to H_2$ is M-P invertible if and only if $\mathrm{im}(f) \subseteq H_2$ is closed.

R. Hagen, S. Roch & B. Silbermann **C*-algebras and numerical analysis.**.

Example

(FHILB, $\dagger$) is a Moore-Penrose dagger category where we this time use SVD on linear operators to construct the M-P inverse.

Examples 2

Example

Let k be a field, and let $\bullet_k$ be the category with one object and whose maps are elements of k. Then $(\bullet_k, \dagger)$ is a Moore-Penrose dagger category where:

$$x^\dagger = x \qquad\qquad x^\circ = \begin{cases} x^{-1} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

A Moore-Penrose dagger category with only one object is precisely a $\star$-**regular monoid**

M. P. Drazin **Regular semigroups with involution.** (1979)

Example

$(\text{REL}, \dagger)$ is NOT a Moore-Penrose dagger category but we can precisely characterize which maps do have M-P inverses:

- A relation $R \subseteq X \times Y$ has a M-P inverse if and only if R is a partial isometry (which are better known as **difunctional relations**). In this case the M-P inverse is the converse relations $R^\circ = R^\dagger \subseteq Y \times X$.

The same is true for allegories.

Examples 3

Example

Let $(\mathbb{X}, \dagger)$ be dagger category where $\mathbb{X}$ is a groupoid (so every map is an isomorphism – not necessarily unitary!). Then $(\mathbb{X}, \dagger)$ is a Moore-Penrose dagger category where $f^\circ = f^{-1}$.

Example

An **inverse category** is a dagger category $(\mathbb{X}, \dagger)$ where:

$$ff^\dagger f = f \qquad\qquad ff^\dagger gg^\dagger = gg^\dagger ff^\dagger$$

Every inverse category $(\mathbb{X}, \dagger)$ is a Moore-Penrose dagger category where $f^\circ = f^\dagger$.

$(\mathsf{PINJ}, \dagger)$ is an inverse category where for a partial injection $f : X \to Y$, $f^\dagger : Y \to X$ is defined as $f^\dagger(y) = x$ if $f(x) = y$ and is undefined otherwise.

More generally, for any **restriction category**, its subcategory of partial isomorphisms is a Moore-Penrose dagger category.

Cockett, R. and Lack, S. **Restriction Categories**.

M-P Inverses and SVD

- SVD can be used to compute Moore-Penrose inverses of complex matrices. We generalize this to dagger categories.

- But before that we consider compact SVD, which is often easier to compute than full SVD:

$$A = R \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_k \end{bmatrix} S^\dagger$$

  where R is a $n \times k$ matrix, S an $m \times k$ matrix, and $R^\dagger R = S^\dagger S = I_k$.

$$A^\circ = S \begin{bmatrix} \frac{1}{d_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{d_k} \end{bmatrix} R^\dagger$$

- Why CSVD instead of SVD? Because CSVD can be defined in arbitrary dagger category, while SVD needs dagger biproducts.

Compact SVD

Definition

In a dagger category, a **generalized compact singular value decomposition** (GCSVD) of a map $f : A \to B$ is a triple $(r : A \to X, d : X \to Y, s : Y \to B)$, where:

- r is a coisometry, so $r^{\dagger} r = 1_X$
- d is an isomorphism
- s is an isometry, so $s s^{\dagger} = 1_Y$

such that $f = rds$.

Lemma

If $f : A \to B$ has a GCSVD $(r : A \to X, d : X \to Y, s : Y \to B)$, then f is M-P invertible with M-P inverse $f^{\circ} = s^{\dagger} d^{-1} r^{\dagger}$.

Can we go from M-P inverses to CSVD? Answer: YES – using dagger idempotent splitting.

Dagger Idempotent Splitting

Definition

In a dagger category $(\mathbb{X}, \dagger)$, a **dagger idempotent** $e : A \to A$ is an idempotent which is self-adjoint, $ee = e = e^\dagger$.

A dagger idemoptent is said to **†-split** if there exists a map $r : A \to X$ such that:

$$rr^\dagger = e \qquad\qquad r^\dagger r = 1_X$$

P. Selinger, **Idempotents in dagger categories**. (2008)

If a map f has a M-P inverse, then ff° and $f^\circ f$ were both †-idempotents.

Definition

In a dagger category $(\mathbb{X}, \dagger)$, a map f is **Moore-Penrose split** (M-P split) if f has a M-P inverse f° and the †-idempotents ff° and $f^\circ f$ †-split.

Proposition

In a dagger category $(\mathbb{X}, \dagger)$, a map f has a GCSVD if and only if f is M-P split.

Neat observation!

In proving that GCSVD = M-P split, you actually show that there is a way to see M-P invertible maps as actual isomorphisms!

Definition

A **dagger idempotent complete category** is a dagger category $(\mathbb{X}, \dagger)$ such that all $\dagger$-idempotents $\dagger$-split.

Every dagger category $(\mathbb{X}, \dagger)$ embeds into a dagger idempotent complete category via the dagger version of the Karoubi envelope, $(\mathsf{Split}_\dagger(\mathbb{X}), \dagger)$.

Lemma

A map $f : A \to B$ in a dagger category $(\mathbb{X}, \dagger)$ has a M-P inverse if and only if there exists $\dagger$-idempotents $e_1 : A \to A$ and $e_2 : B \to B$ such that $f : (A, e_1) \to (B, e_2)$ is an isomorphism in $(\mathsf{Split}_\dagger(\mathbb{X}), \dagger)$.

Theorem

In a dagger idempotent complete category, a map is M-P invertible if and only if it has a GCSVD.

SVD

Definition

A dagger category $(\mathbb{X}, \dagger)$ has **finite †-biproducts** if $\mathbb{X}$ has finite biproducts such that the adjoints of the projections are the injections, that is, $\pi_j^\dagger = \iota_j$.

Definition

In a dagger category $(\mathbb{X}, \dagger)$ with finite †-biproducts, a **generalized singular value decomposition** (GSVD) of a map $f : A \to B$ is a triple of maps $(u : A \to X \oplus Z, d : X \to Y, v : Y \oplus W \to B)$:

- u and v are unitary, so isomorphisms such that $u^{-1} = u^\dagger$ and $v^{-1} = v^\dagger$;
- d is an isomorphism

and such that $f = u(d \oplus 0)v$.

Proposition

If $f : A \to B$ has a GSVD $(u : A \to X \oplus Z, d : X \to Y, v : Y \oplus W \to B)$. Then f is M-P split where $f^\circ := v^\dagger(d^{-1} \oplus 0)u^\dagger$.

Using matrix representation, we see that this recaptures SVD for complex matrices:

$$f = u \begin{bmatrix} d & 0 \\ 0 & 0 \end{bmatrix} v \qquad\qquad f^\circ = v^\dagger \begin{bmatrix} d^{-1} & 0 \\ 0 & 0 \end{bmatrix} u^\dagger$$

Dagger Kernels

How do we go from M-P inverses to GSVD? Answer: dagger kernels!

Definition

In a dagger category $(\mathbb{X}, \dagger)$ with a zero object, a map $f : A \to B$ has a **$\dagger$-kernel** if f has a kernel $k : \ker(f) \to A$ such that k is an isometry. A **dagger kernel category** is a dagger category with a zero object such that every map has a dagger kernel.

B. Jacobs & C. Heunen **Quantum logic in dagger kernel categories**. (2010)

Proposition

In a dagger kernel category $(\mathbb{X}, \dagger)$ with finite $\dagger$-biproducts and negatives[a]*, a map f has a GSVD if and only if f is M-P split.*

[a] You can do this without negatives, but extra mild assumptions need to hold.

The CSVD is $(u : A \to X \oplus \ker(f), d : X \to Y, v : Y \oplus \ker(f^\dagger) \to B)$, where X is the splitting of ff° and Y is the splitting of $f^\circ f$.

Theorem

In a dagger kernel category $(\mathbb{X}, \dagger)$ that is $\dagger$-idempotent complete and which has finite $\dagger$-biproducts and negatives, a map f is M-P invertible if and only if f has a GSVD.

Polar Decomposition

N. Higham, **Functions of matrices: theory and computation**. (2008)

nicely explains how M-P inverses can play a role in the polar decomposition of complex matrices

Definition

In a dagger category $(\mathbb{X}, \dagger)$, for a map $f : A \to B$ a **Moore-Penrose polar decomposition** (M-P PD) is a pair of maps $(u : A \to B, h : B \to B)$ where:

- u is a partial isometry;
- h is a positive map, so $h = gg^\dagger$, which is M-P invertible

such that $f = uh$ and $u^\dagger u = hh^\circ$.

Definition

In a dagger category $(\mathbb{X}, \dagger)$, a M-P invertible positive map $p : A \to A$ has a **Moore-Penrose square root** (M-P square root) if there exists a M-P invertible positive map $\sqrt{p} : A \to A$ such that $\sqrt{p}\sqrt{p} = p$.

P. Selinger, **Idempotents in dagger categories**. (2008)

Proposition

In a dagger category $(\mathbb{X}, \dagger)$, a map f has a M-P PD if and only if f is M-P invertible and $f^\dagger f$ has a M-P square root.

For $\Rightarrow$: $f^\circ = h^\circ u^\dagger$. For $\Leftarrow$: the M-P PD is $(f(\sqrt{f^\dagger f})^\circ, \sqrt{f^\dagger f})$.

Concluding Thoughts

- M-P inverses should also be considered in relation to other dagger structures, such as:
  - Dagger Monads
  - Dagger Limits
  - Dagger Compact Closed

- More examples of Moore-Penrose dagger categories! Some candidates:
  - Portions of the ZX-Calculus
  - PROPs with weights on strings
  - Graphical Linear Algebra

- Applications of Moore-Penrose inverses?
  - Electrical Circuits

    S. Campbell & C. Meyer **Generalized Inverses of Linear Transformations**.

  - Trace

    M. Bartha **Quantum Turing Automata**.

  - Conditionals in Markov Categories

    T. Fritz **A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics**.

  - Any more ideas?

A Wise Man Climbs Mt.Fuji Once, Only A Fool Would Climb it Twice

By this logic, Robin is a wise man.... I'm a certified fool (baka)!

Aug 15 2022

Aug 30 2022

HOPE YOU ENJOYED MY TALK!

THANK YOU FOR LISTENING!

MERCI!

Email: `js.lemay@mq.edu.au`

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