AXIOMS FOR THE CATEGORY OF: HILBERT SPACES & LINEAR CONTRACTIONS

arXiv:2211.02688

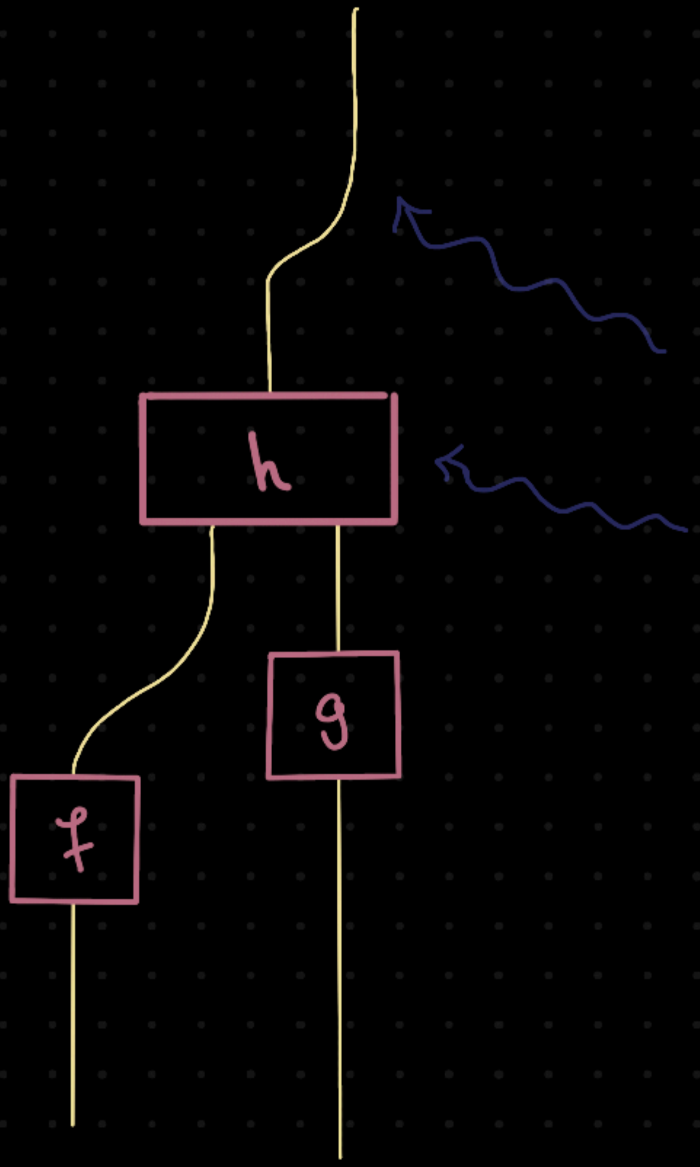
NESTA VAN DER SCHAAF
JOINT WITH: CHRIS HEUNEN & ANDRE KORNELL

QPL 2023
19 JULY 2023

CONTENTS.

- MOTIVATION (QM)
- THE QUESTION (RECONSTRUCTION)
- RECAP OLD RESULT (HEUNEN + KORNELL)
- PROOF STRATEGY
- SCALARS
- NEW AXIOMS
- MAIN CONSTRUCTION
- THE THEOREM

PHYSICS AS PROCESSES.

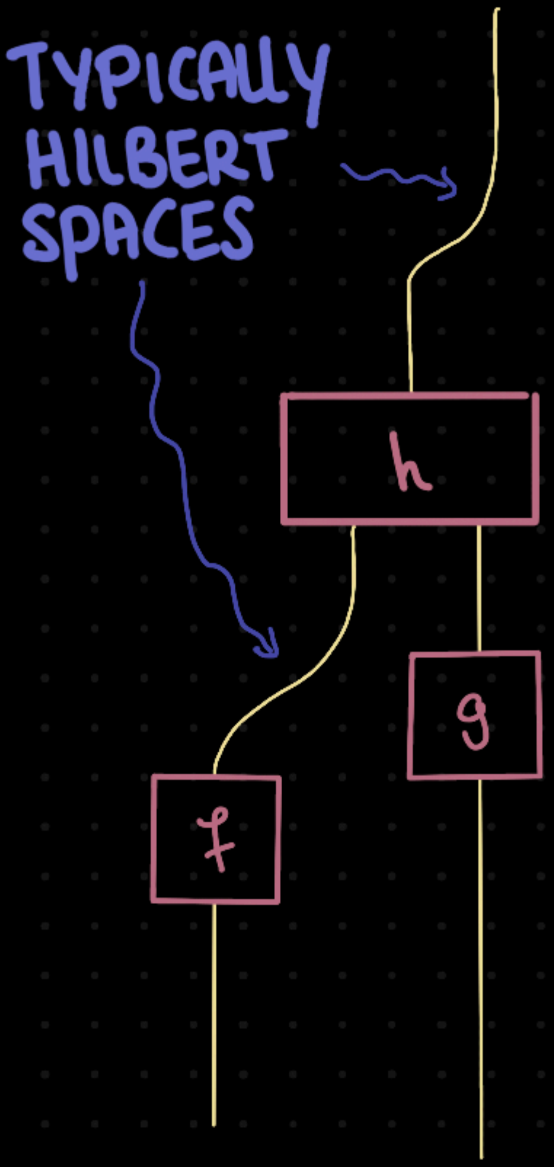


MONOIDAL CATEGORIES:

OBJECTS = PHYSICAL SYSTEMS
ARROWS = PHYSICAL PROCESSES

COMPOSITION = SEQUENTIAL COMPOSITION
TENSOR = PARALLEL COMPOSITION

PHYSICS AS PROCESSES.

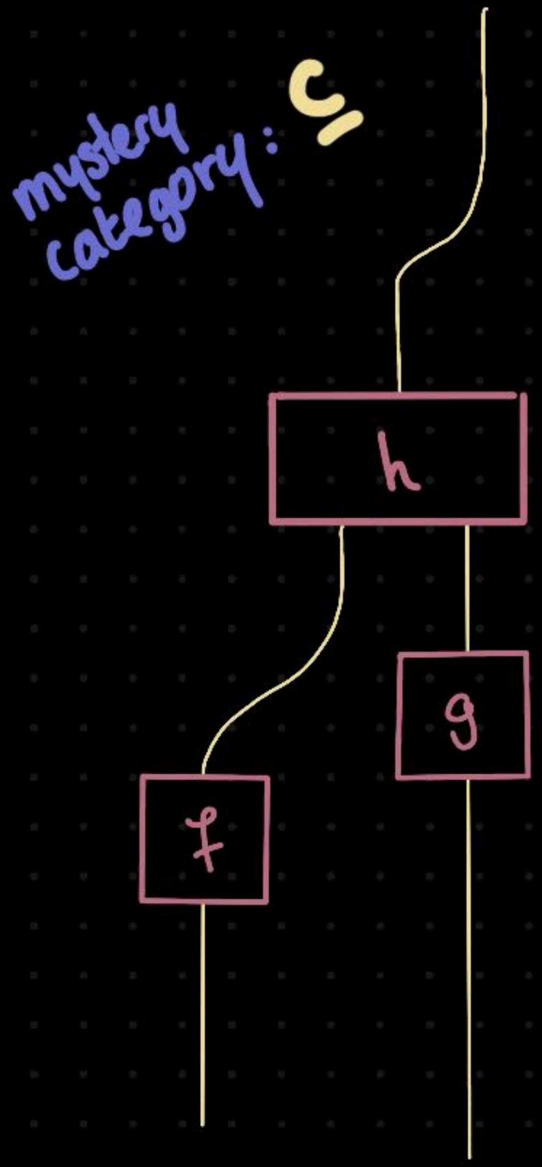


CATEGORICAL QUANTUM MECHANICS: QM

HILB
- objects : HILBERT SPACES
- arrows : BOUNDED LINEAR MAPS

CON
- objects : IDEM.
- arrows : LINEAR f s.t.: $\|f\| \leq 1$

POSING THE QUESTION.



HOW CAN WE TELL OUR PROCESSES ARE QUANTUM ?

WHAT ARE PROPERTIES OF $\underline{C}$ SUCH THAT:

$$\underline{C} \simeq \underline{QM} \ ?$$

(AS DAGGER MONOIDAL CATS.)

ANSWERS?

- HILB: HEUNEN + KORNELL, PNAS 2022 Vol. 119 No. 9, arXiv: 2109.07418.

THIS TALK

- CON: HEUNEN + KORNELL + vdS, arXiv: 2211.02688.

- ONGOING: (NOT BY ME!!) $\mathbf{HILB}_A$, $\underbrace{\mathbf{FHILB},\ \mathbf{UNITARY}}_{\text{HEUNEN + DIMEGLIO}\ldots}$, ...

RECAP OF THE OLD RESULT.

HEUNEN + KORNELL
PNAS 2022 Vol. 119 No. 9
arXiv: 2109.07418.

THE AXIOMS :

(A) $\mathbf{C}$ IS DAGGER MONOIDAL

$\dagger : \mathbf{C}^{op} \longrightarrow \mathbf{C}$, id.-on-obj., idempotent, $id_H^\dagger = id_H$.

Monoidal str. whose coherence morphisms are $\dagger$-iso., $f^{-1} = f^\dagger$.

(B) I IS A SIMPLE SEPARATOR

SIMPLE: I has two subobjects.

MON. SEPARATOR: if $\forall I \xrightarrow{h} H \; \forall I \xrightarrow{k} K$: $f \circ (h \otimes k) = g \circ (h \otimes k)$, then $f = g$.

(C) $\mathbf{C}$ HAS $\dagger$ - BIPRODUCTS

$\mathbf{C}$ has a zero obj. 0. Coproducts:

$$\begin{tikzcd} H \arrow[dr, "i"'] & & K \arrow[dl, "j"] \\ & H \oplus K & \end{tikzcd}$$

$\dagger$-monomorphisms ← $f^\dagger \circ f = id.$

$j^\dagger \circ i = 0$

(D) $\mathbf{C}$ HAS $\dagger$ - EQUALISERS

All equalisers exist, and they are $\dagger$-monomorphisms

(E) $\dagger$-MONOS ARE $\dagger$-KERNELS

Any $\dagger$-mono f is a $\dagger$-kernel, i.e. an equaliser:

$$N \xrightarrow{f} K \overset{\exists}{\underset{0}{\rightrightarrows}} H$$

(F) $\mathbf{C}$ HAS DIRECTED COLIMITS OF $\dagger$-MONOS

RECAP OF THE OLD RESULT.

HEUNEN + KORNELL
PNAS 2022 Vol. 119 No. 9
arXiv: 2109.07418.

THEOREM. IF $\mathbf{C}$ SATISFIES AXIOMS (A) — (F), THEN:

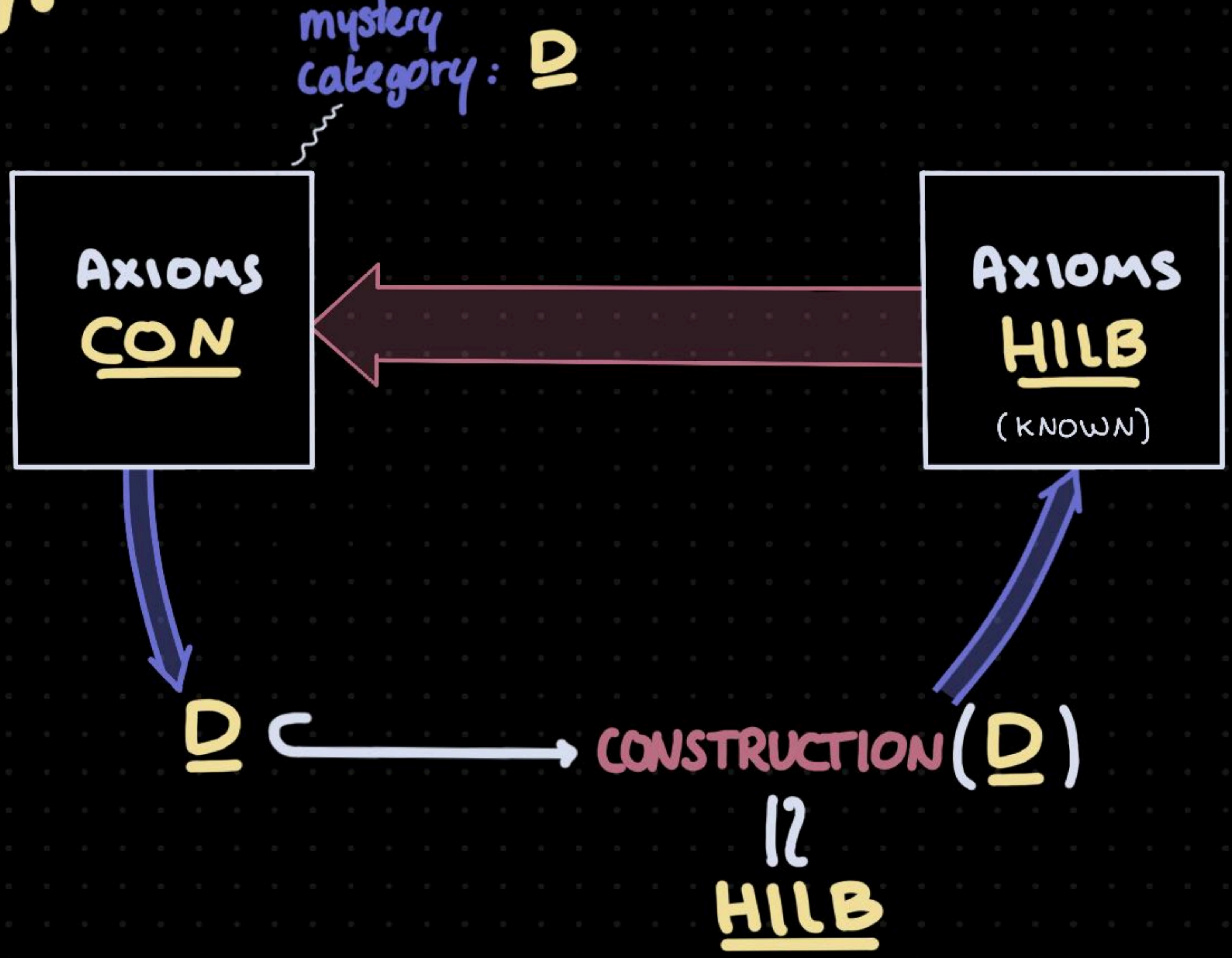
$$\mathbf{C} \simeq \mathbf{HILB}.$$

(AS DAGGER MONOIDAL CATS.)

THE STRATEGY.

SO WE NEED TO FIGURE OUT:

HOW ARE <u>HILB</u> AND <u>CON</u> RELATED?



THE IDEA:

EVERY BOUNDED MAP IS OF THE FORM:

$$\underbrace{f}_{\text{HILB}} = \underbrace{z}_{\text{SCALAR}} \cdot \underbrace{g}_{\text{CON}}$$

SCALARS.

IN A MONCAT. $\underline{\mathcal{C}}$ THERE ARE SCALARS:

$$z : I \longrightarrow I$$

THEY FORM A COMMUTATIVE MONOID UNDER COMPOSITION

SCALARS z AND MORPHISMS f CAN BE MULTIPLIED:

$$\begin{tikzcd}
H \arrow[r, "z \bullet f"] \arrow[d, "\wr"] & K \\
I \otimes H \arrow[r, "z \otimes f"'] & I \otimes K \arrow[u, "\wr"']
\end{tikzcd}$$

FROM FUNCTORIALITY OF $\otimes$ WE GET:

$$\begin{aligned}
& z \bullet (g \circ f) \\
& \qquad \| \\
& (z \bullet g) \circ f \qquad \text{“SHUFFLING”} \\
& \qquad \| \\
& g \circ (z \bullet f)
\end{aligned}$$

SCALARS IN <u>HILB</u> & <u>CON</u>.

$\mathbb{C}$

complex numbers

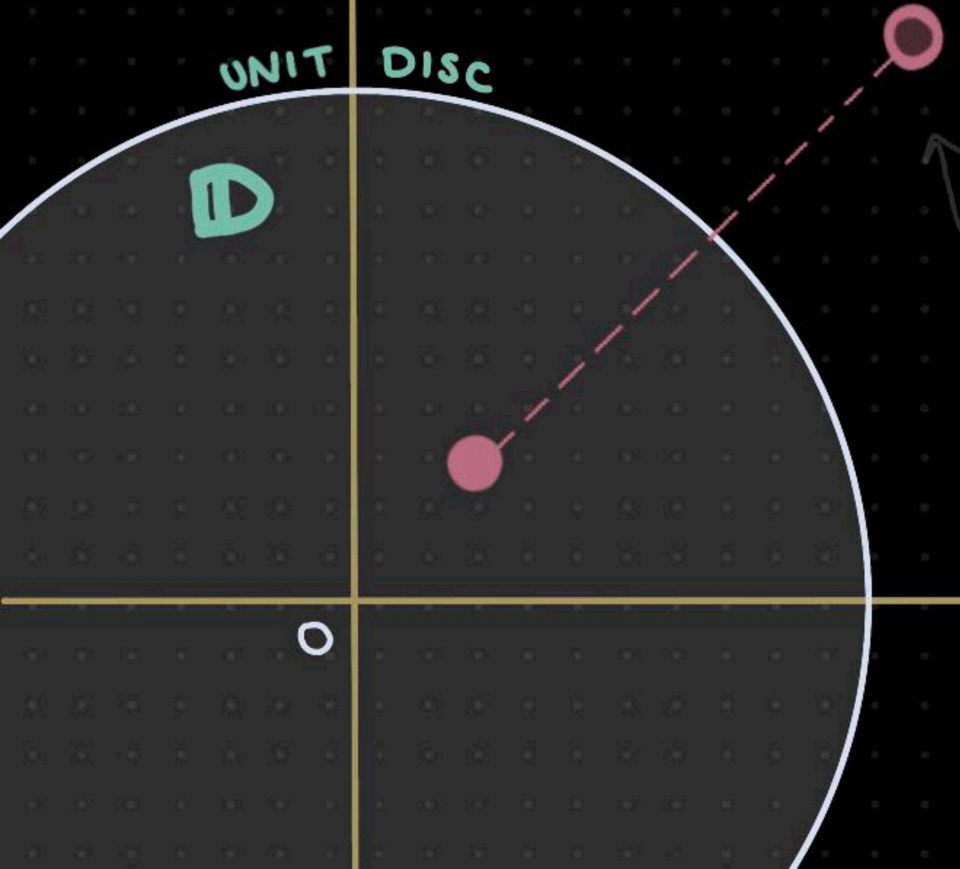
WE FIND:

$\underline{\text{HILB}}(\mathbb{C},\mathbb{C}) \cong \mathbb{C}$ AND $\underline{\text{CON}}(\mathbb{C},\mathbb{C}) \cong \mathbb{D}$.

$$f = z \cdot g$$

$\mathbb{C}$ $\mathbb{D}$

SCALAR



PROBLEM: $\mathbb{D} \cdot \mathbb{D} \neq \mathbb{C}$

<u>THE IDEA</u>: CONSTRUCTION(—) ADDS FORMAL INVERSES FOR ALL SCALARS.

THE CONSTRUCTION.

GIVEN $\begin{cases} \text{MON CAT} : \underline{D} \\ \text{SCALARS} : \mathbb{D} := \underline{D}(I, I) \end{cases}$

THERE IS A CAT.

$$\text{CONSTRUCTION}(\underline{D}) = \underline{D}(\mathbb{D}^{-1})$$

WITH:

OBJECTS: SAME AS $\underline{D}$

ARROWS: OF THE FORM

$$H \xrightarrow{[t/z]} K$$

UNDER EQUIVALENCE RELATION:

$$(t/z) \sim (t'/z') \iff z' \bullet t = z \bullet t'$$

THE CONSTRUCTION.

THE CONSTRUCTION IS UNIVERSAL:

PROPOSITION.

$$\begin{tikzcd} \underline{D} \arrow[r, hook] \arrow[dr, "F"'] & \underline{D}(\mathbb{D}^{-1}) \arrow[d, dashed, "\exists!"] \\ & \underline{C} \end{tikzcd}$$

AND : CONSTRUCTION$(\underline{\text{CON}}) \cong \underline{\text{HILB}}$.

THE NEW AXIOMS.

(1) $\underline{D}$ IS A †- CAT.

(2) $\underline{D}$ IS A †- RIG CAT.

Two †-monoidal structures: $(\otimes, I)$ and $(\oplus, 0)$ such that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, $(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger$ and $\otimes$ distributes over $\oplus$.

ALMOST BIPRODUCTS

(3) $(\oplus, 0)$ IS AFFINE

0 is initial, and hence a zero obj. This gives natural:

$$\mathrm{inl}_{HK} : H \longrightarrow H \oplus K$$

$$\mathrm{inr}_{HK} : K \longrightarrow H \oplus K$$

(4) inl, inr ARE JOINTLY EPIC

$$\left.\begin{array}{l} f \circ \mathrm{inl} = g \circ \mathrm{inl} \\ f \circ \mathrm{inr} = g \circ \mathrm{inr} \end{array}\right\} \Longrightarrow f = g$$

(5) THERE IS MIXTURE

$\exists s : I \longrightarrow I \oplus I$ with $\mathrm{inl}^\dagger \circ s \neq 0 \neq \mathrm{inr}^\dagger \circ s$

UNIT AXIOMS

(6) I IS †- SIMPLE

(7) I IS A $\otimes$-SEPARATOR

†-AXIOMS

(8) $\underline{D}$ HAS ALL †- EQUALISERS

(9) †- MONOS ARE †- KERNELS

(10) SUBOBJECTS ARE DETERMINED BY POSITIVE MAPS

$s = t$ as subobj. iff $s \circ s^\dagger = t \circ t^\dagger$

(11) $\underline{D}$ HAS ALL DIRECTED COLIMITS

THE CONSTRUCTION.

AFTER A LOT OF DETAILS... :

THEOREM. IF $\underline{D}$ SATISFIES AXIOMS (I) — (II) THEN $\underline{D}(\mathbb{D}^{-1})$ SATISFIES AXIOMS (A) – (F), SO:

$$\begin{tikzcd} \underline{D} \arrow[r, hook] & \underline{D}(\mathbb{D}^{-1}) \arrow[d, phantom, sloped, "\cong"] \\ & \underline{\mathrm{HILB}} \end{tikzcd}$$

HENCE $\underline{D} \subseteq \underline{\mathrm{HILB}}$.
WE ARE "JUST" LEFT TO SHOW:

$$\underline{D} = \underline{\mathrm{CON}}.$$

TOWARDS THE THEOREM.

HENCE $\underline{D} \subseteq \underline{\text{HILB}}$.
WE ARE "JUST" LEFT TO SHOW:
$\underline{D} = \underline{\text{CON}}$.

actual complex numbers

LEMMA. $\mathbb{D} = \{ z \in \mathbb{C} : |z| \leq 1 \}$.

LEMMA. $\underline{D}(H,H) = \{ t \in \underbrace{\underline{\text{HILB}}(H,H)}_{C^*\text{-algebra}} : \| t \| \leq 1 \}$.

(BY RUSSO-DYE-GARDNER:
$t = \frac{1}{n}(u_1 + \cdots + u_n)$
unitary)

LEMMA. IN GENERAL:

$$\underline{D}(H,K) = \underline{\text{CON}}(H,K)$$

(BY POLAR DECOMPOSITION)

THE THEOREM.

THEOREM. IF $\underline{D}$ SATISFIES AXIOMS (I) — (III), THEN:

$$\underline{D} \simeq \underline{CON} .$$

THE THEOREM.

THEOREM. IF $\underline{D}$ SATISFIES AXIOMS (I) — (III), THEN:

$$\underline{D} \simeq \underline{CON}.$$

Thanks very much for your attention!