Simpler Complete Equational Theories for Quantum Circuits with Ancillae or Partial Trace

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Mocqua / Loria

QPL23

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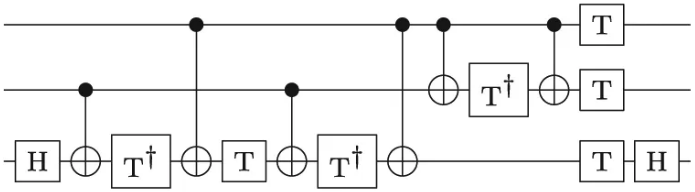
Quantum Circuits



Quantum Circuits

D. Deutsch. Quantum computational networks. Proceedings of the Royal Society of London, A425:73–90, 1989.

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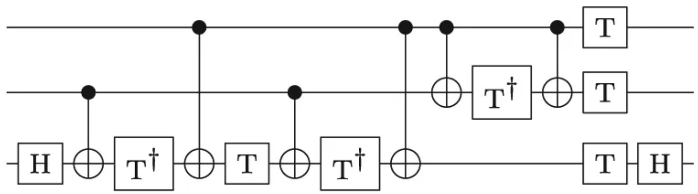

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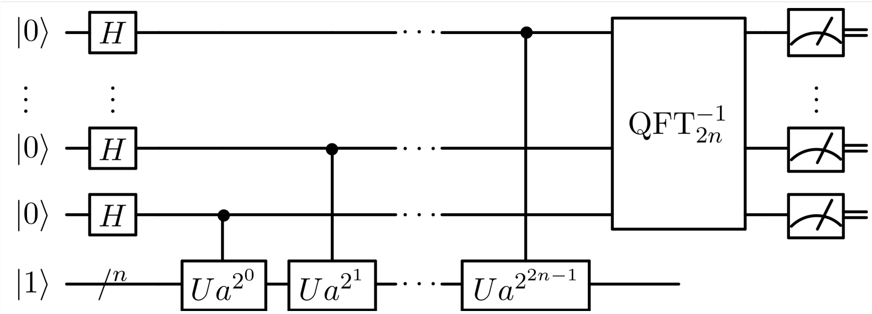
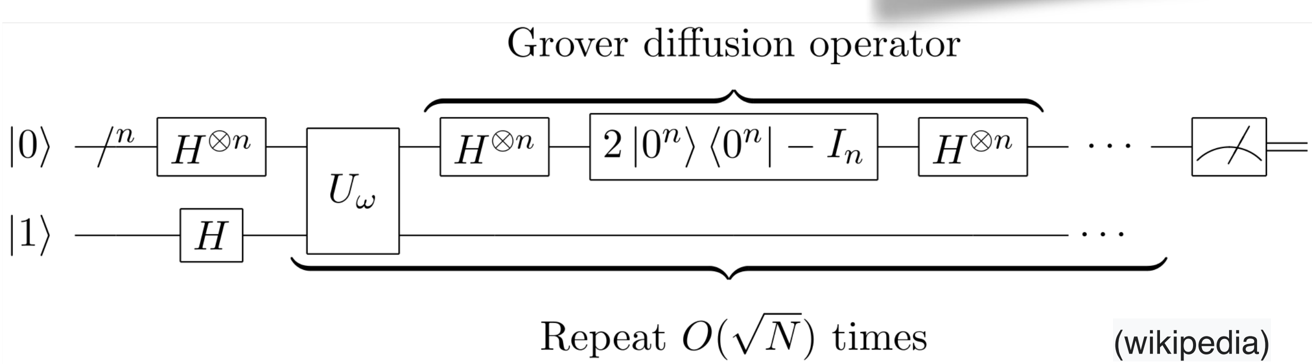


Quantum subroutine in Shor's algorithm (wikipedia)



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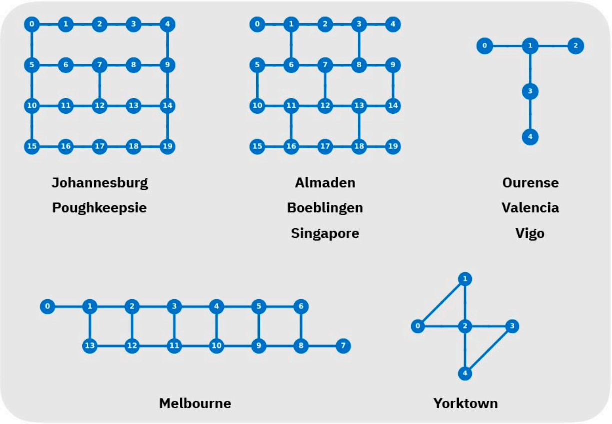
Quantum Circuits

Ubiquitous intermediate language for:

- Resource optimisation (#gates, #T, #CNot…)
- Hardware-constraint satisfaction (primitives, topological constraints, …)
- Fault-tolerant Quantum Computing
- Verification, circuit equivalence testing.

=> Circuit Transformation

IBM's 10 Quantum Device Lineup



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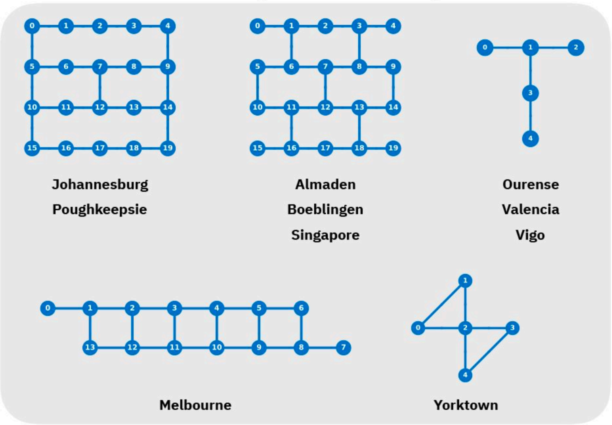

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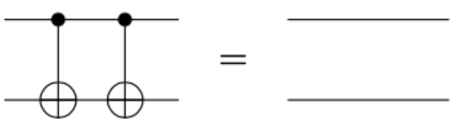
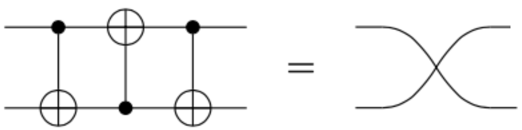
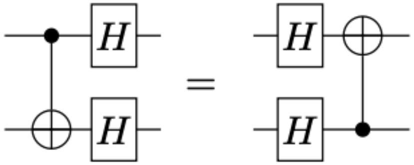
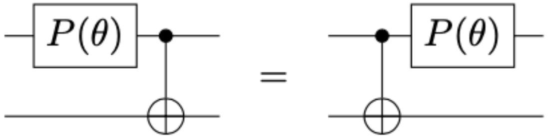
Equational theory, e.g.:









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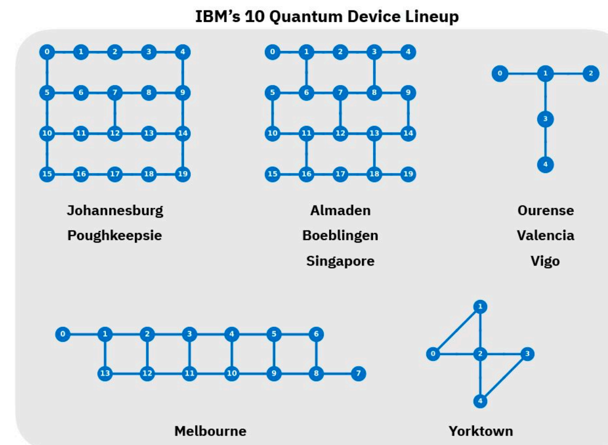

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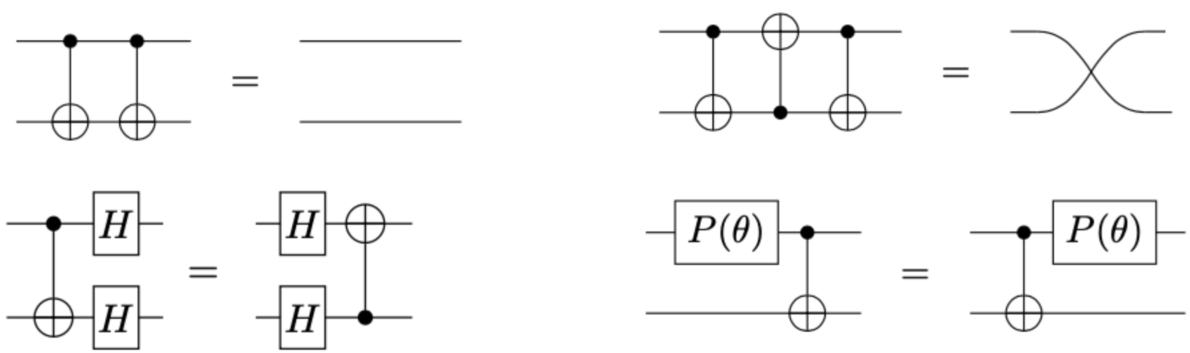
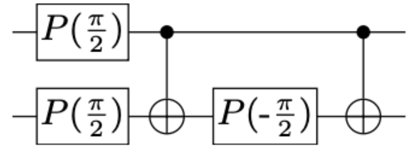


Completeness[1]?

1. if two circuits represent the same unitary, one can be transformed into the other using the equational theory, i.e, all true equations can be derived.

Circuits versus Matrices

Definition. The prop of quantum circuits is generated by $-\boxed{H}-$: $1 \to 1$, $-\boxed{P(\varphi)}-$: $1 \to 1$,  : $2 \to 2$, $\varphi : 0 \to 0$ for any $\varphi \in \mathbb{R}$.



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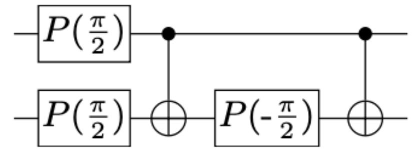
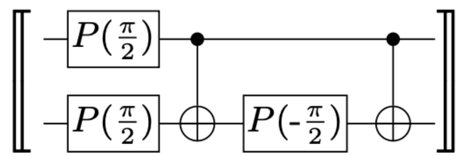


For any quantum circuit C, $[\![C]\!]$ is the corresponding matrix.



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

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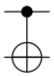
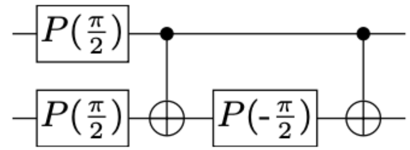
Definition. The prop of quantum circuits is generated by $-\boxed{H}-$: $1 \to 1$, $-\boxed{P(\varphi)}-$: $1 \to 1$,  : $2 \to 2$, φ : $0 \to 0$ for any $\varphi \in \mathbb{R}$.



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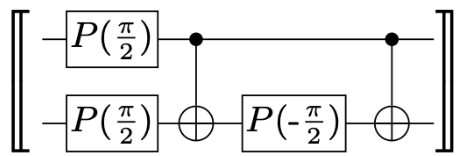

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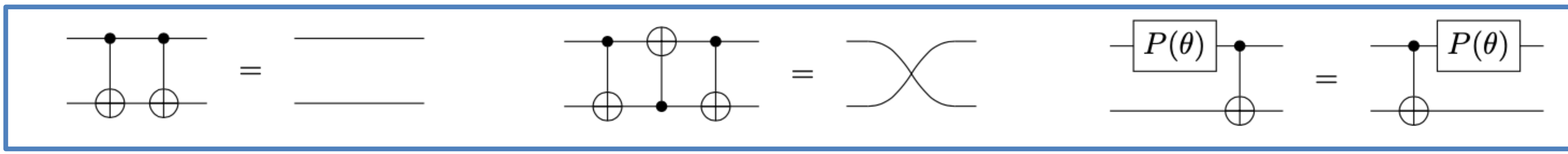
Equational theory, e.g.:

(E_0)



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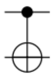
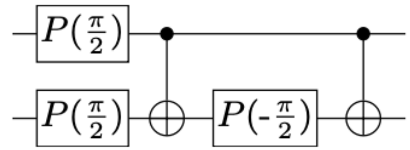
Definition. The prop of quantum circuits is generated by $\boxed{H} : 1 \to 1$, $\boxed{P(\varphi)} : 1 \to 1$,  $: 2 \to 2$, $\varphi : 0 \to 0$ for any $\varphi \in \mathbb{R}$.



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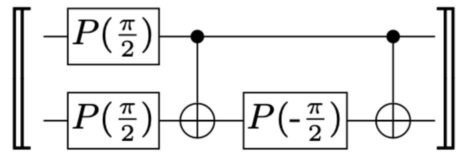

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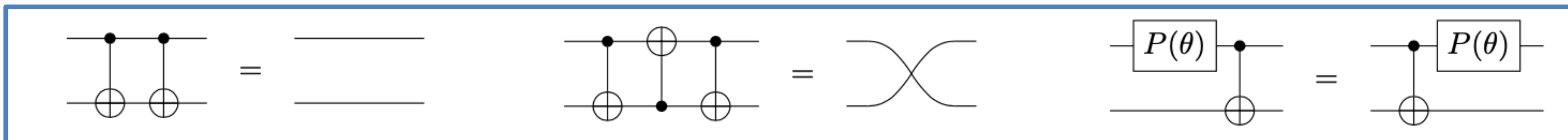

Soundness. If $E \vdash C_0 = C_1$ then $[\![C_0]\!] = [\![C_1]\!]$.

Completeness. If $[\![C_0]\!] = [\![C_1]\!]$ then $E \vdash C_0 = C_1$.

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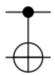
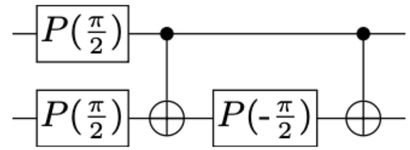
Definition. The prop of quantum circuits is generated by $-\boxed{H}-$: $1 \to 1$, $-\boxed{P(\varphi)}-$: $1 \to 1$,  : $2 \to 2$, $\textcircled{\varphi}$: $0 \to 0$ for any $\varphi \in \mathbb{R}$.



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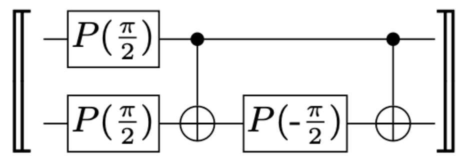

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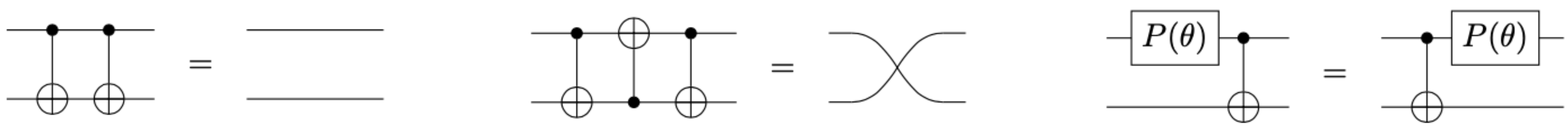

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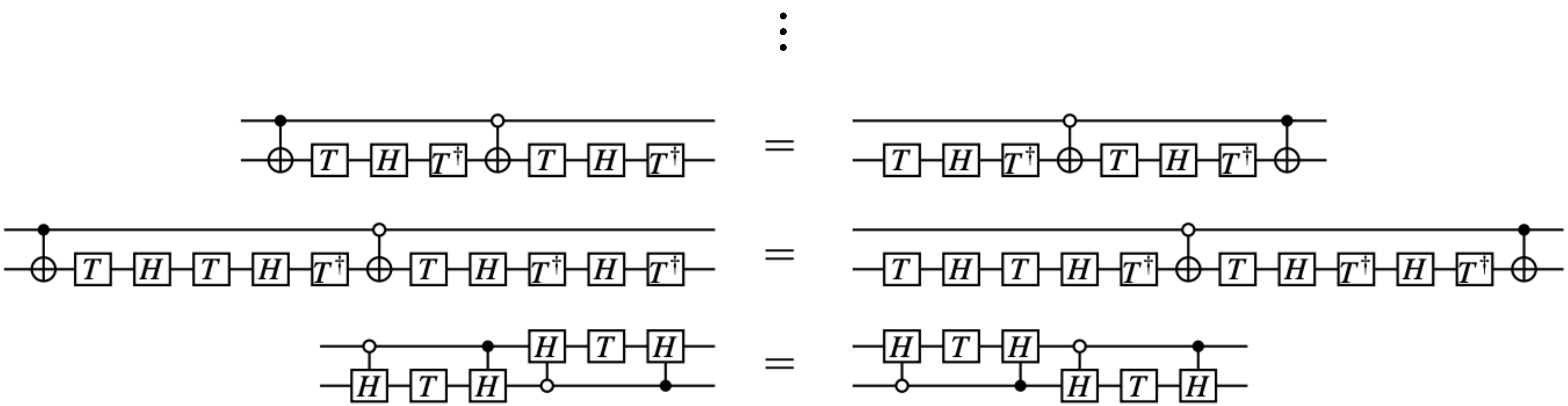
Example. (E_0) is sound but not complete:

QC Completeness

Complete equational theories for non-universal and classically simulatable **fragments**:

- 2-qubit circuits (Clifford+T) [Bian,Selinger'22]



(C18)

(C19)

(C20)

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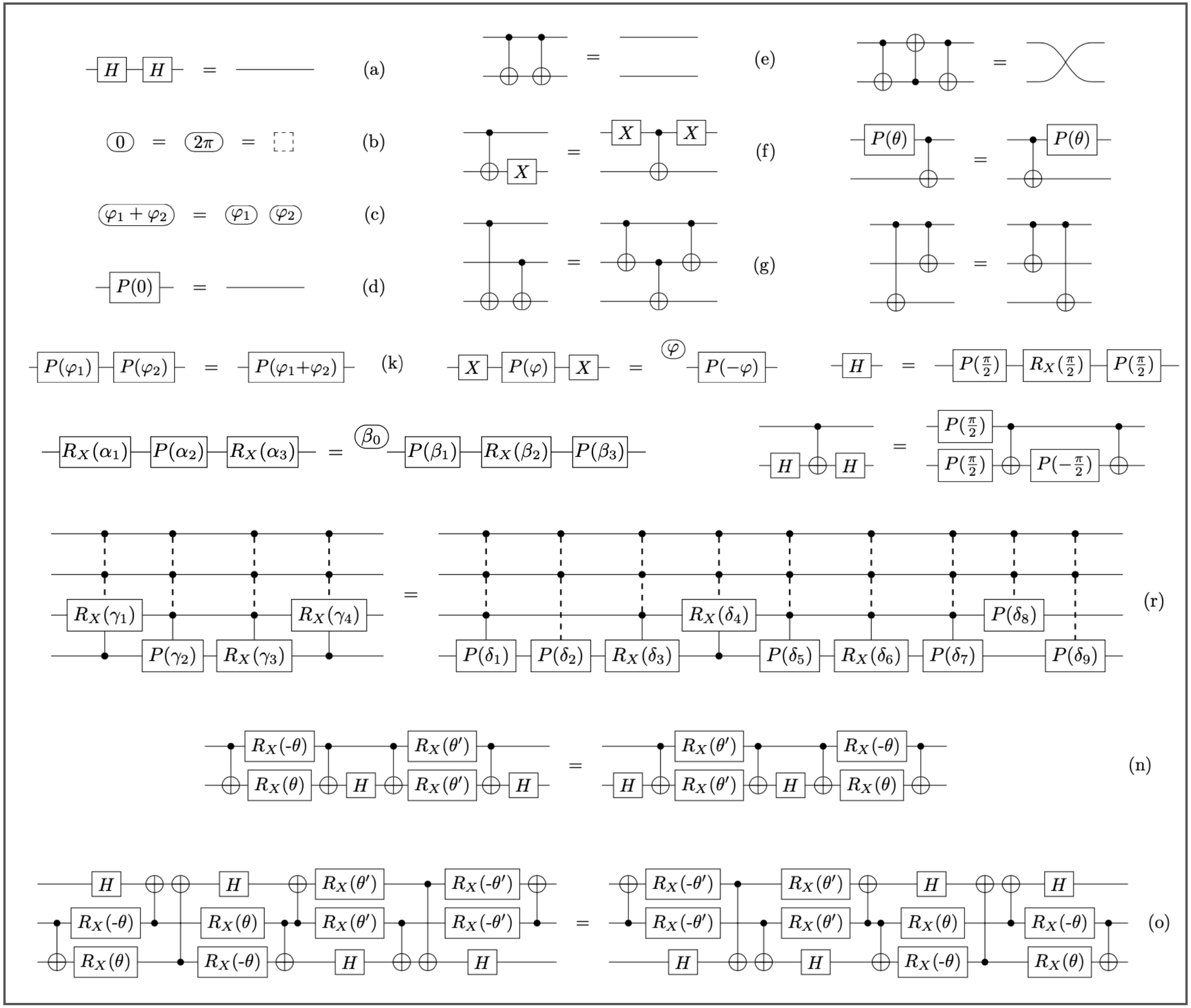
Our contributions:

- Simplifying equational theory for vanilla QC
- Extension to quantum circuits with ancilla and/or discarding

Complete Equational Theory [CHMPV LICS'23]



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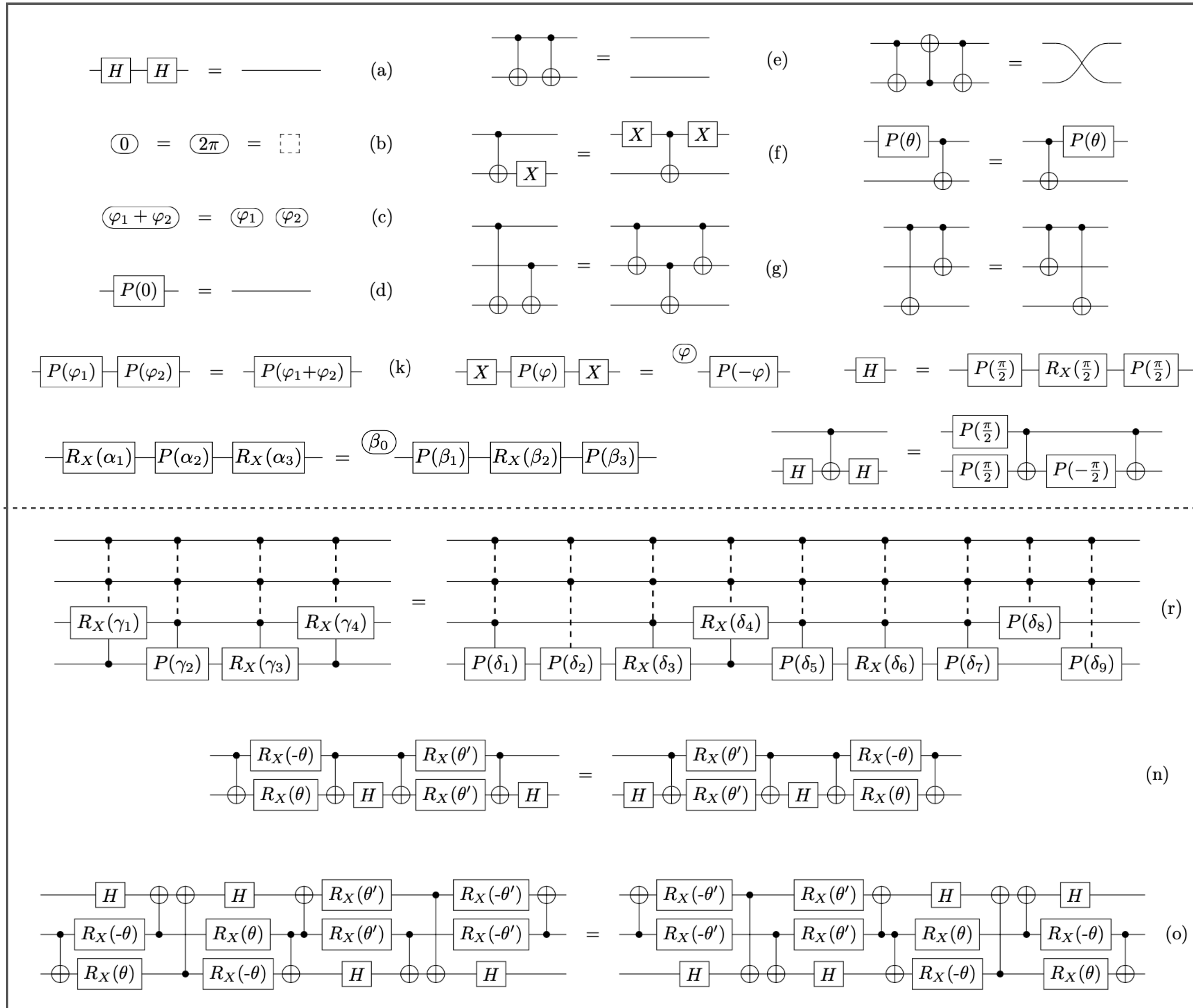

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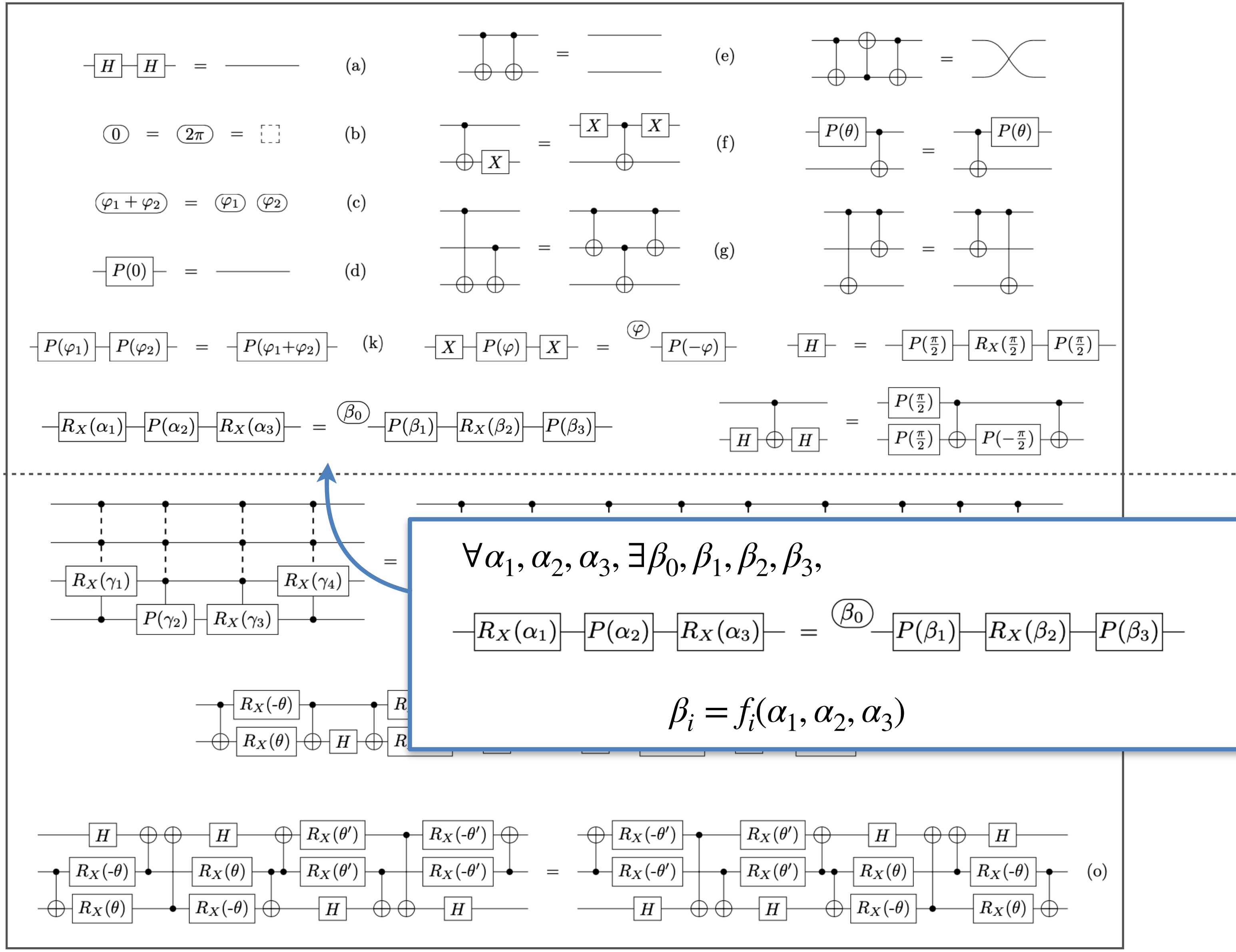

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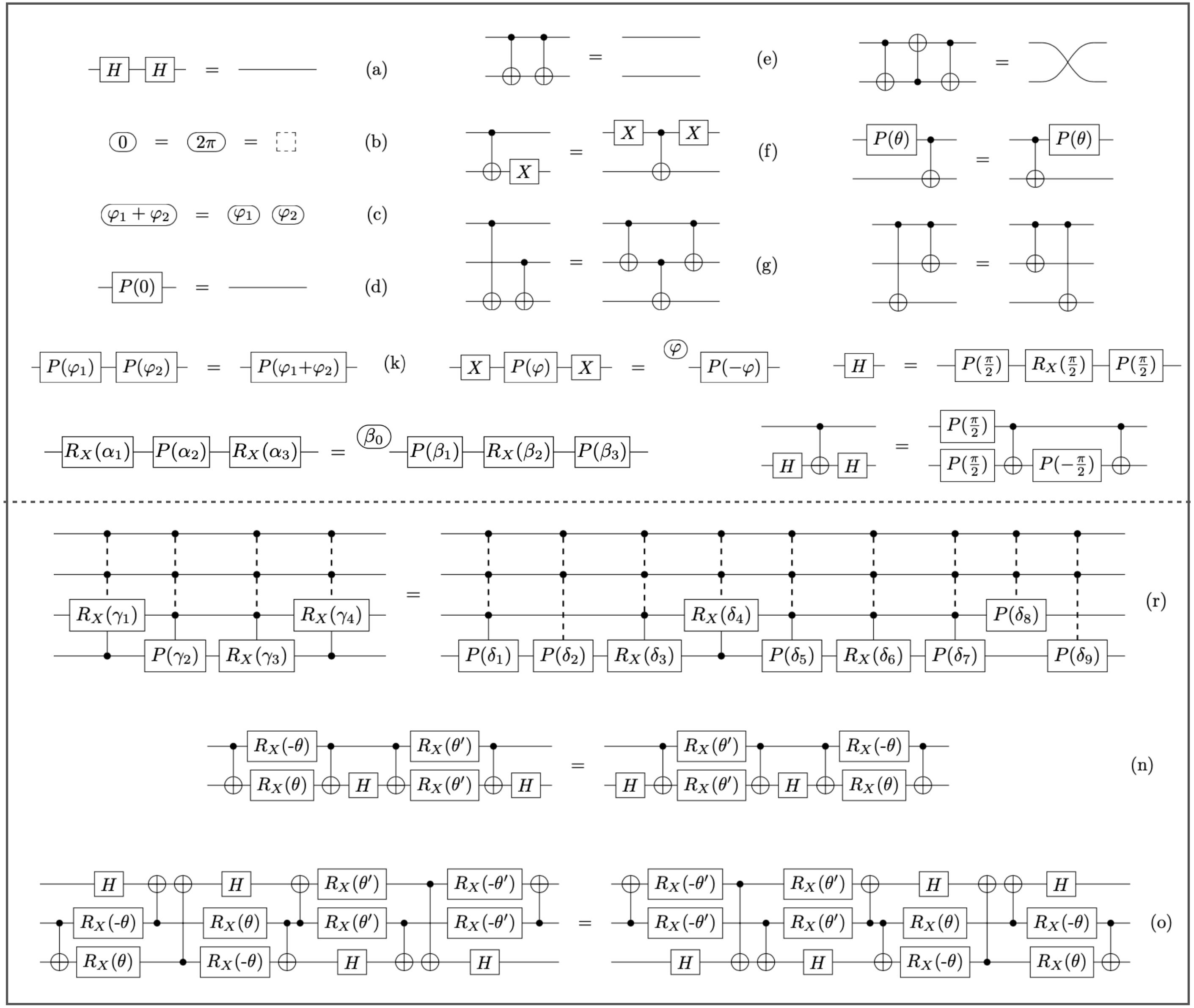

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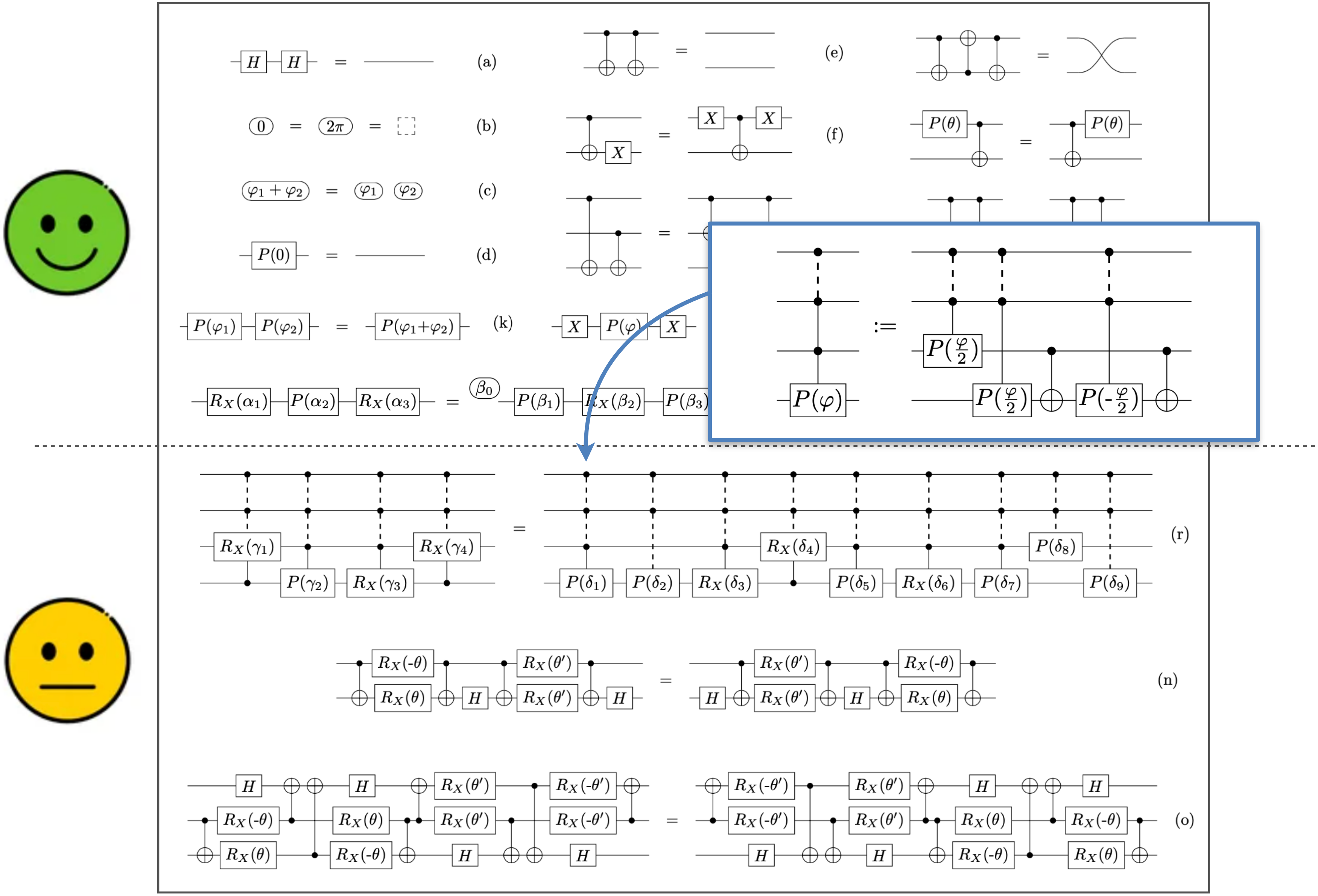

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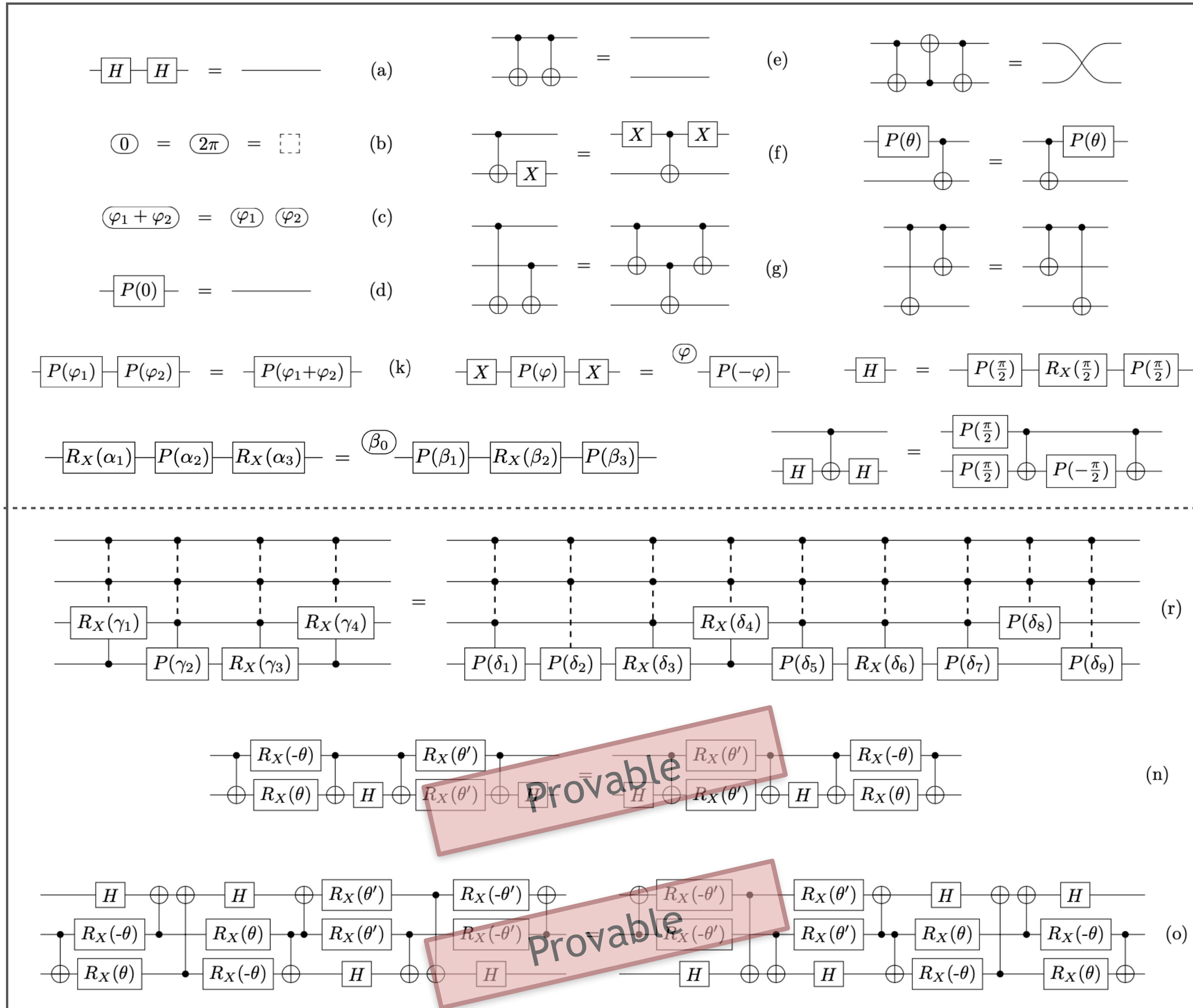

Property [CDPV QPL23]

(n) and (o) can be derived from the other equations, and (r) (slightly) simplified

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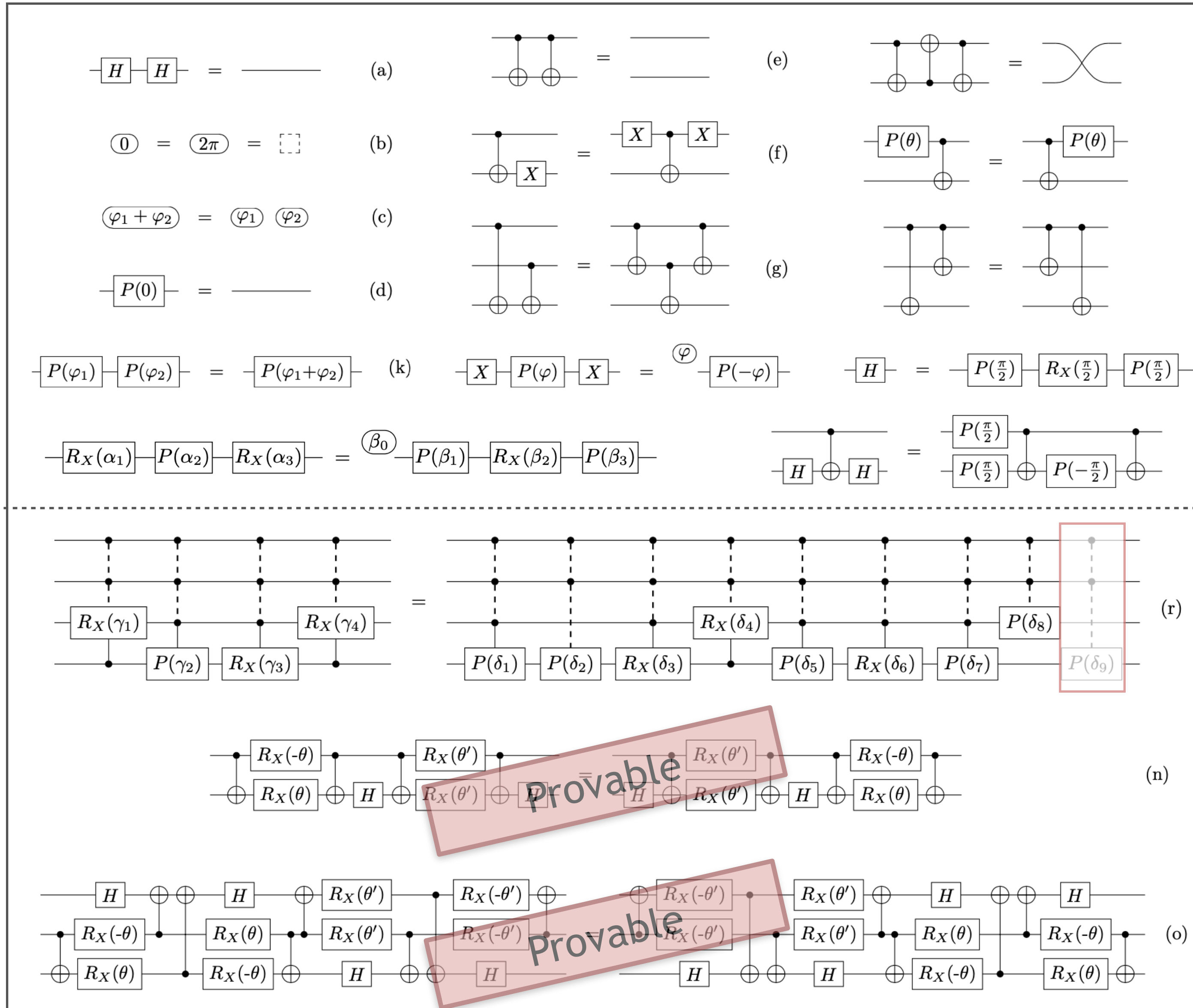

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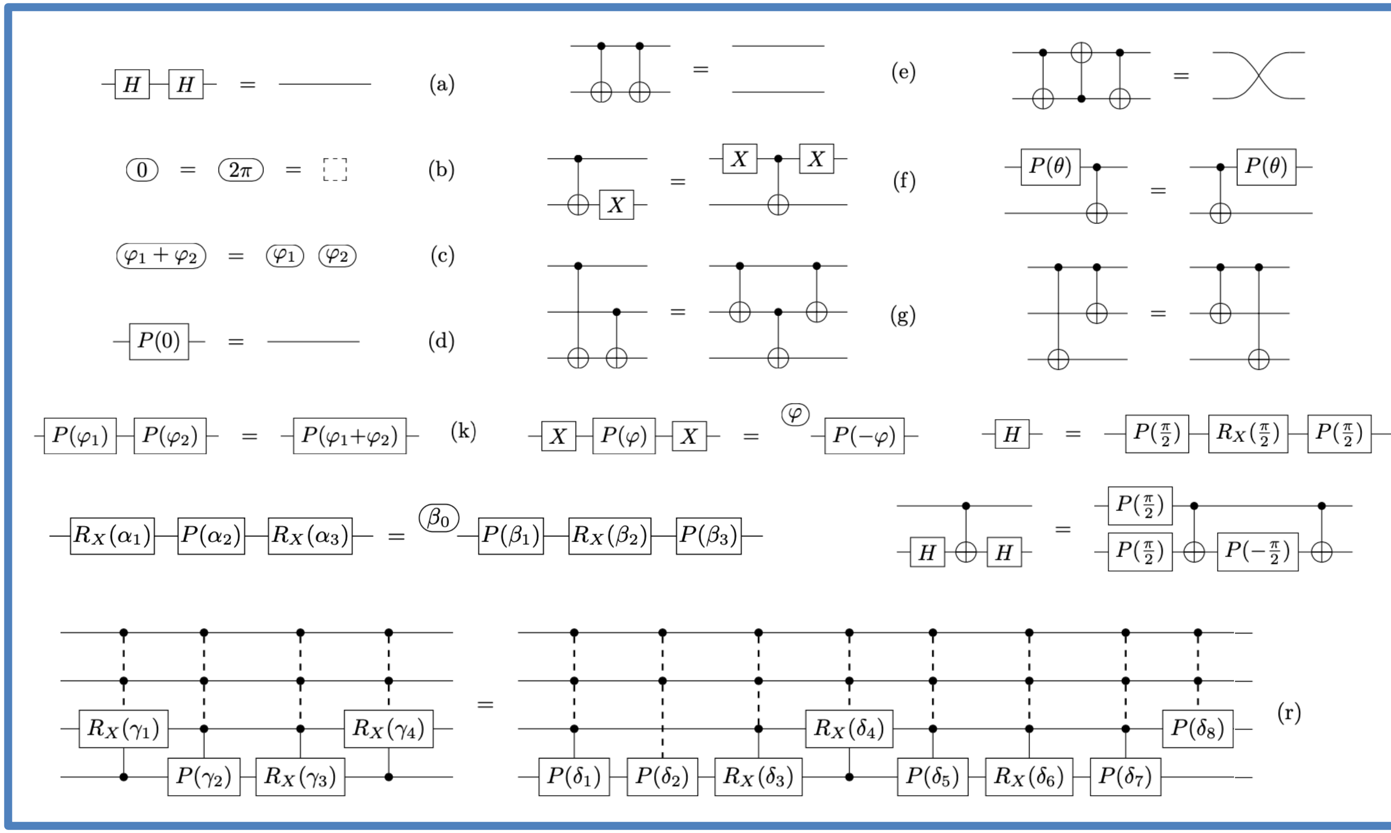
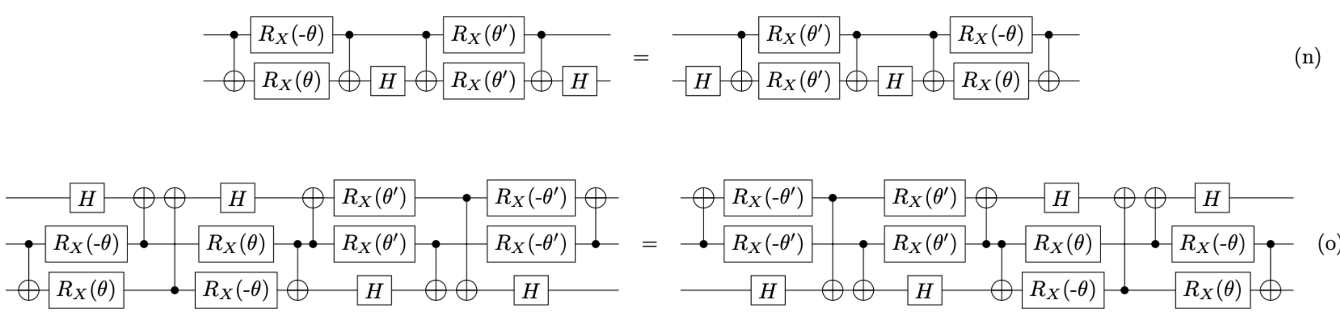
Equational Theory

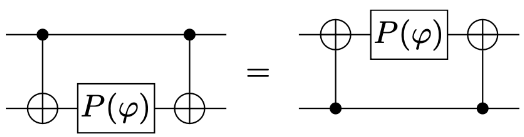
QC





Derivations in QC

- Useful properties, e.g.:

QC $\vdash$ 

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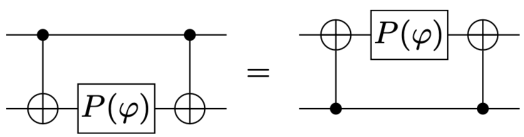
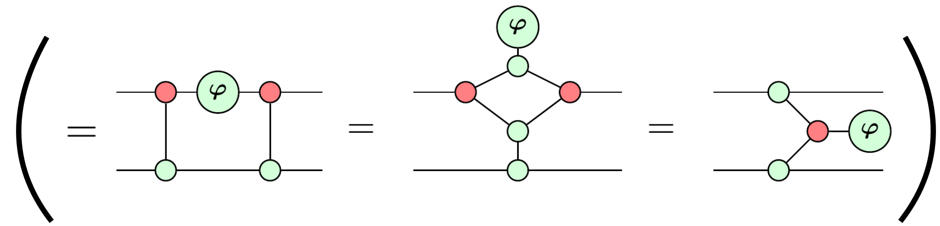
QC $\vdash$ 



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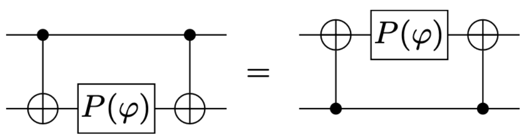
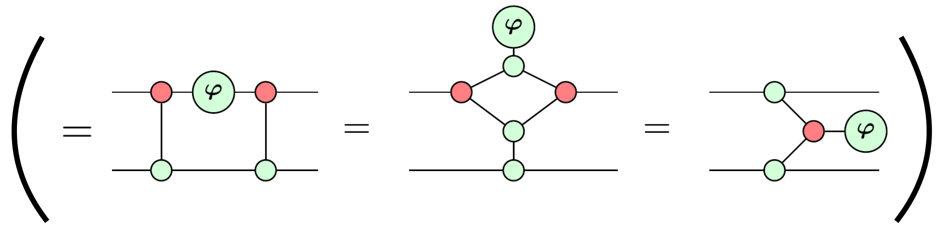
QC ⊢ 



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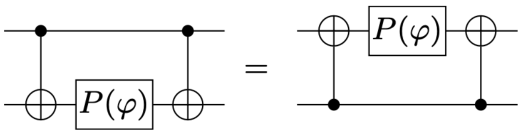
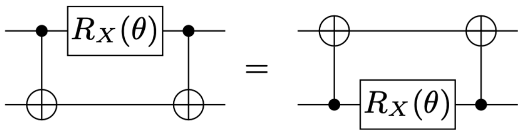
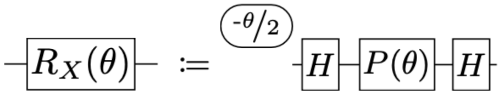
QC $\vdash$ 

QC $\vdash$ 

where 

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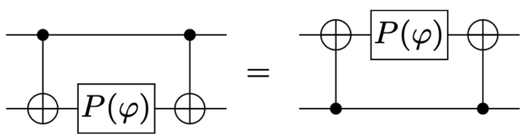
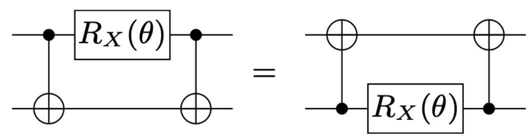
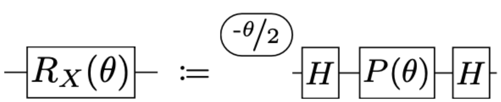
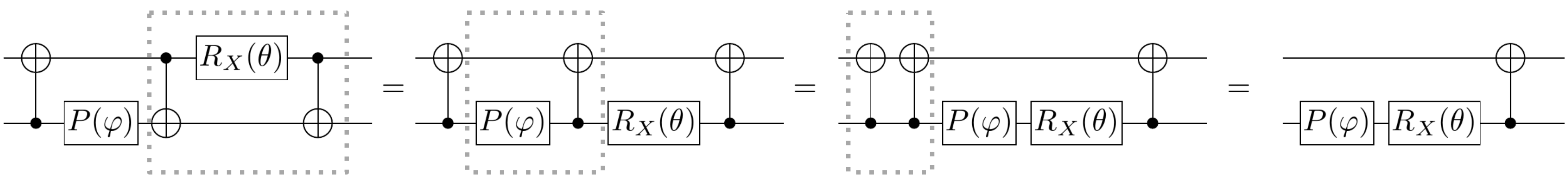
QC $\vdash$ 

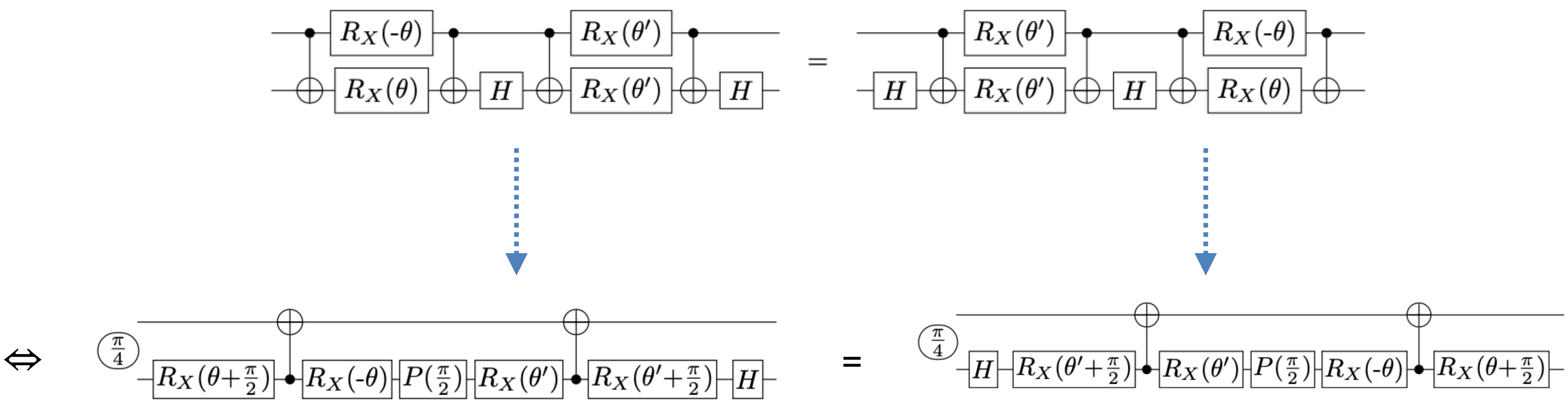
QC $\vdash$ 

where 



Derivations in QC

- Useful properties, e.g.: Phase gadget, Euler decomposition



Derivations in QC

- Useful properties
- **Simplification principal**

Definition For any quantum circuit C, let $C^\dagger$ be the *adjoint* of C inductively defined as $(C_2 \circ C_1)^\dagger := C_1^\dagger \circ C_2^\dagger$; $(C_1 \otimes C_2)^\dagger := C_1^\dagger \otimes C_2^\dagger$; and for any $\varphi \in \mathbb{R}$, $(\textcircled{\varphi})^\dagger := \textcircled{-\varphi}$, $(-\boxed{P(\varphi)}-)^\dagger := -\boxed{P(-\varphi)}-$, and $g^\dagger := g$ for any other generator g.

Proposition. For any circuit C, $\mathrm{QC} \vdash \boxed{C}\,\boxed{C^\dagger} = \overline{}$

Corollary. For any circuits C, C_0, C_1,

$$\mathrm{QC} \vdash \boxed{C_0}\,\boxed{C} = \boxed{C_1} \quad \Leftrightarrow \quad \mathrm{QC} \vdash \boxed{C_0} = \boxed{C_1}\,\boxed{C^\dagger}$$

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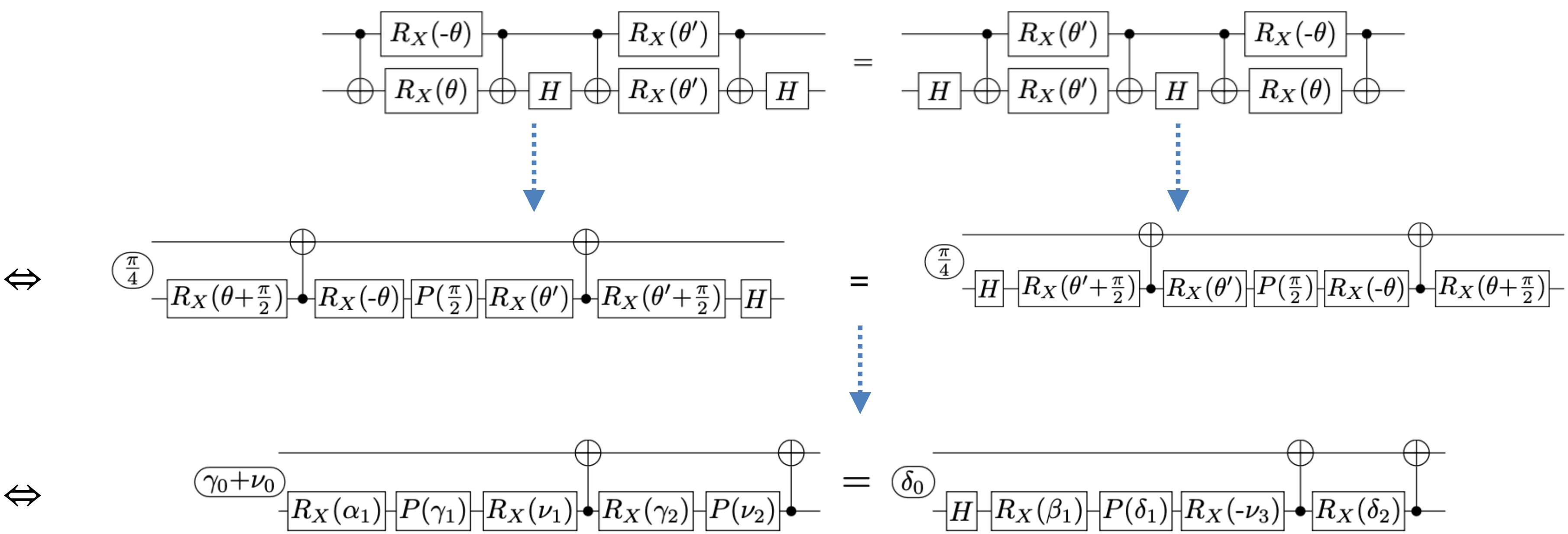
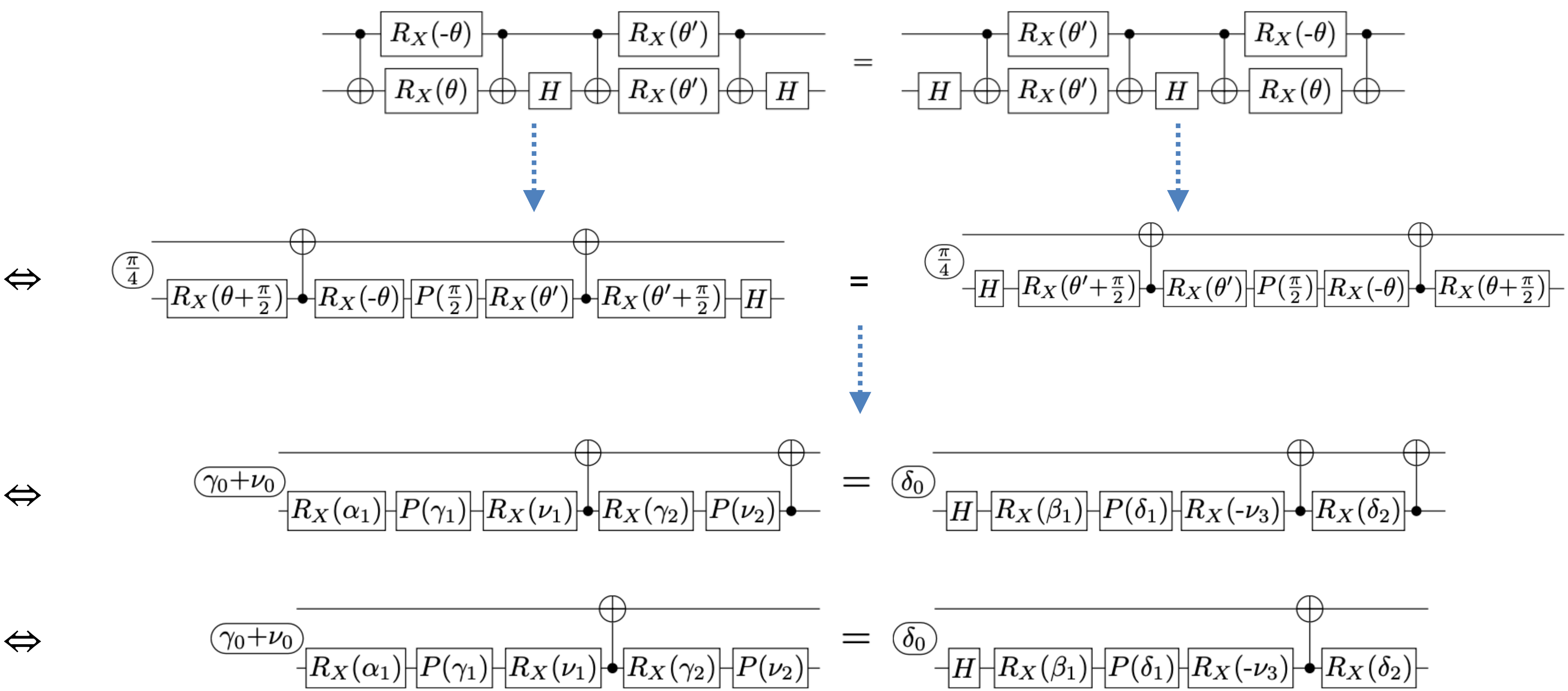
- Useful properties
- **Simplification principal**



Derivations in QC

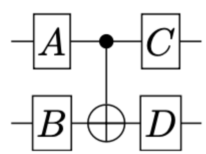
- Useful properties
- **Simplification principal**



Derivations in QC

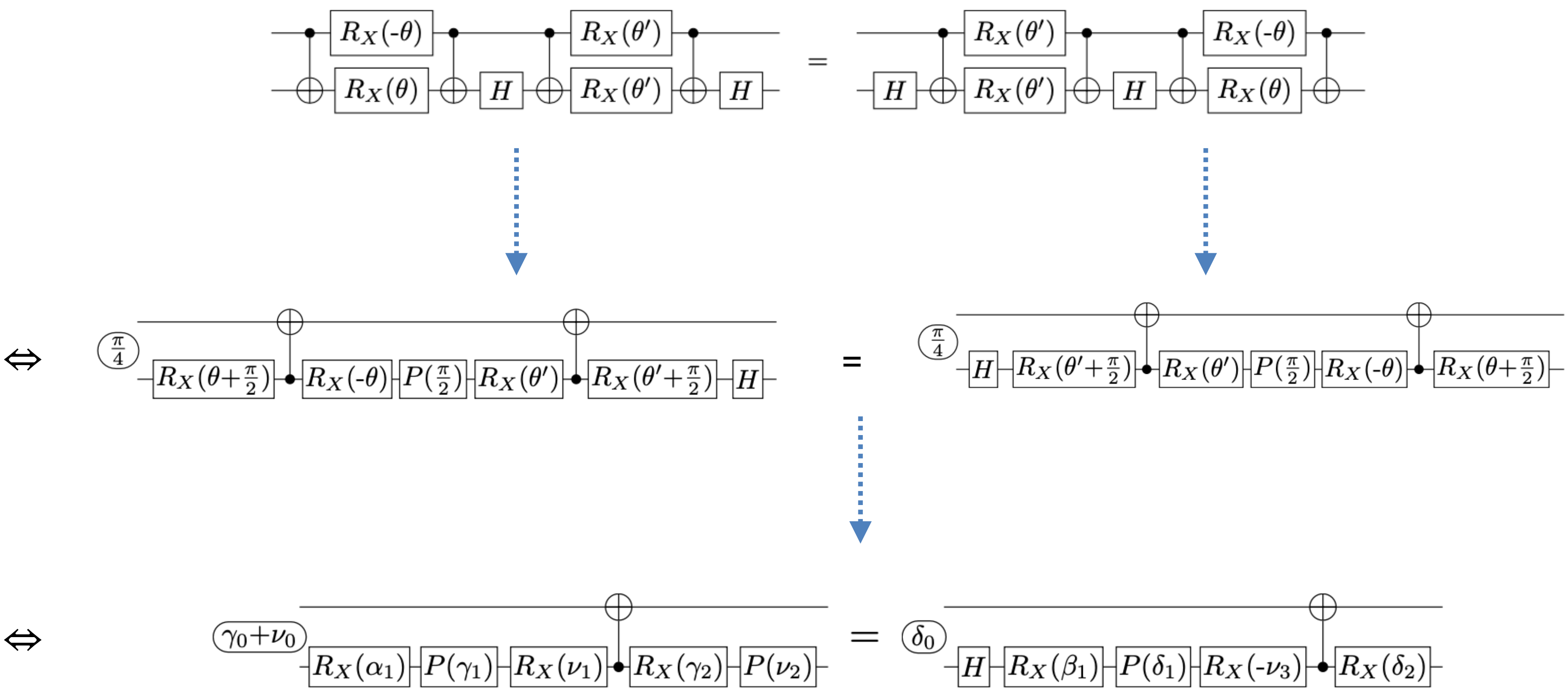
- Useful properties
- Simplification principal
- **1-CNot completeness**

Lemma. QC is complete for circuits containing at most one , i.e. for any quantum circuits $C_1, C_2 \in \mathbf{QC}$ with at most one , if $[\![C_1]\!] = [\![C_2]\!]$ then $\mathrm{QC} \vdash C_1 = C_2$.



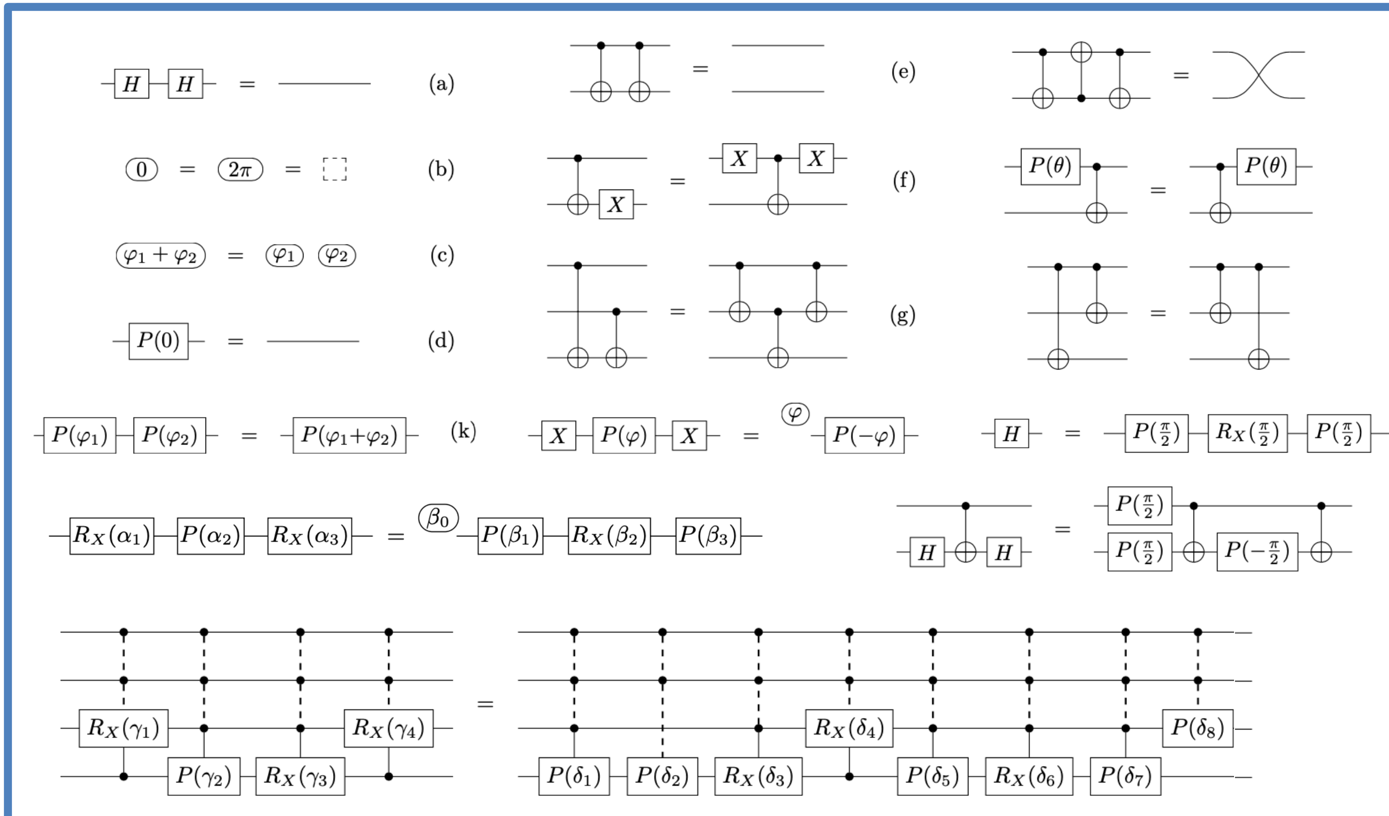
Derivations in QC

- Useful properties
- Simplification principal
- **1-CNot completeness**



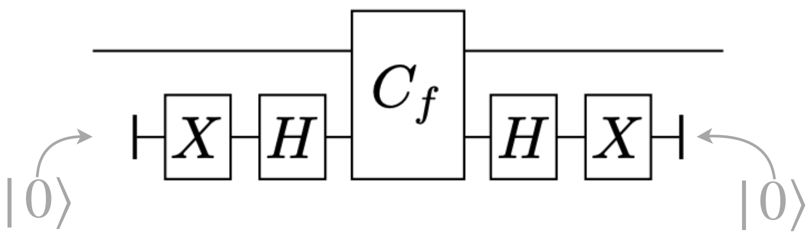
QC completeness

QC



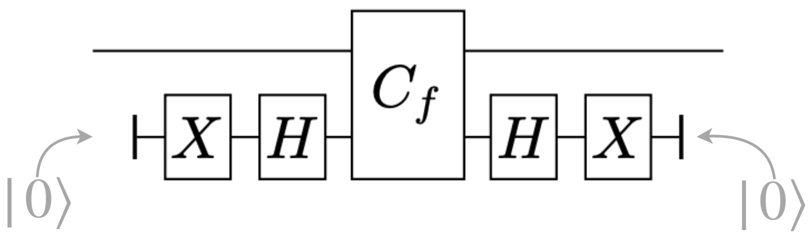
Theorem. QC is complete.

Quantum Circuits with
ancilla and/or trace out

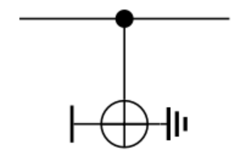


Universal for Isometries

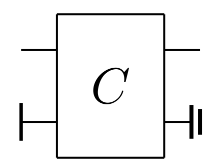
Quantum Circuits with
ancilla and/or trace out



Universal for Isometries



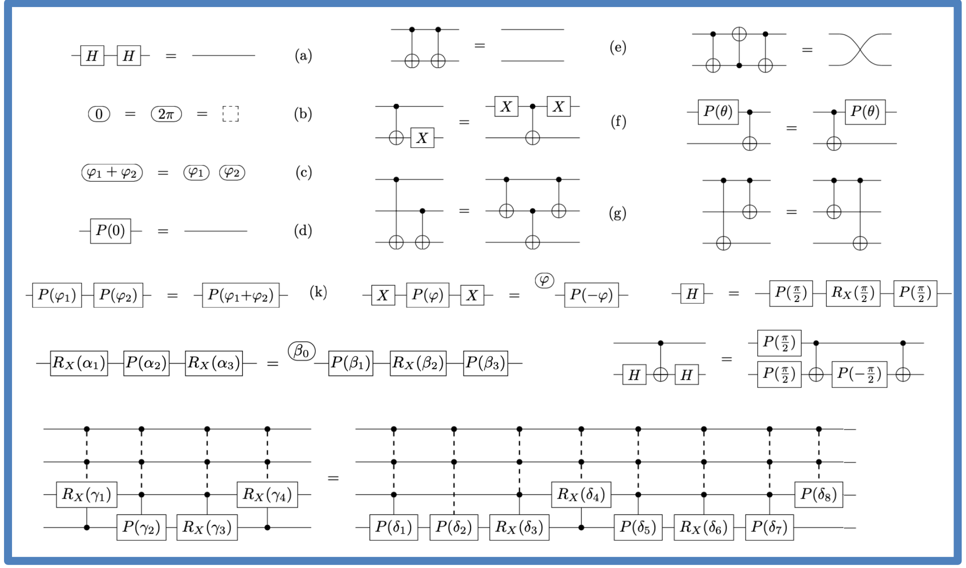
Universal for CPTP maps



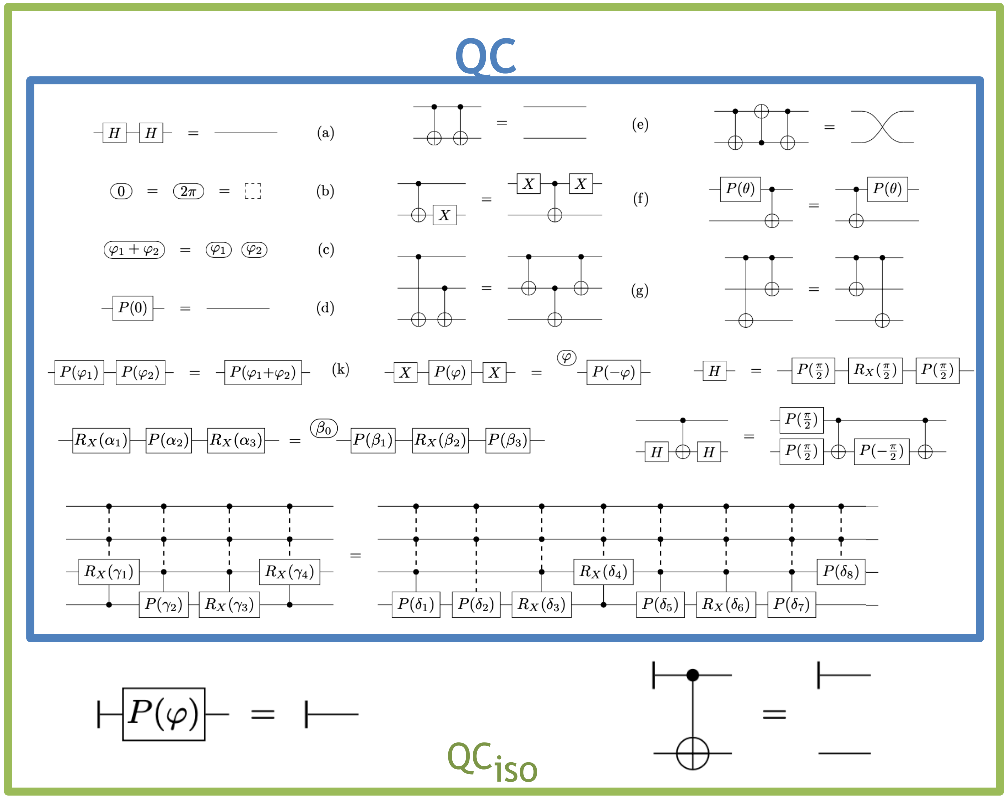
Quantum Circuits with

ancilla and/or trace out

QC



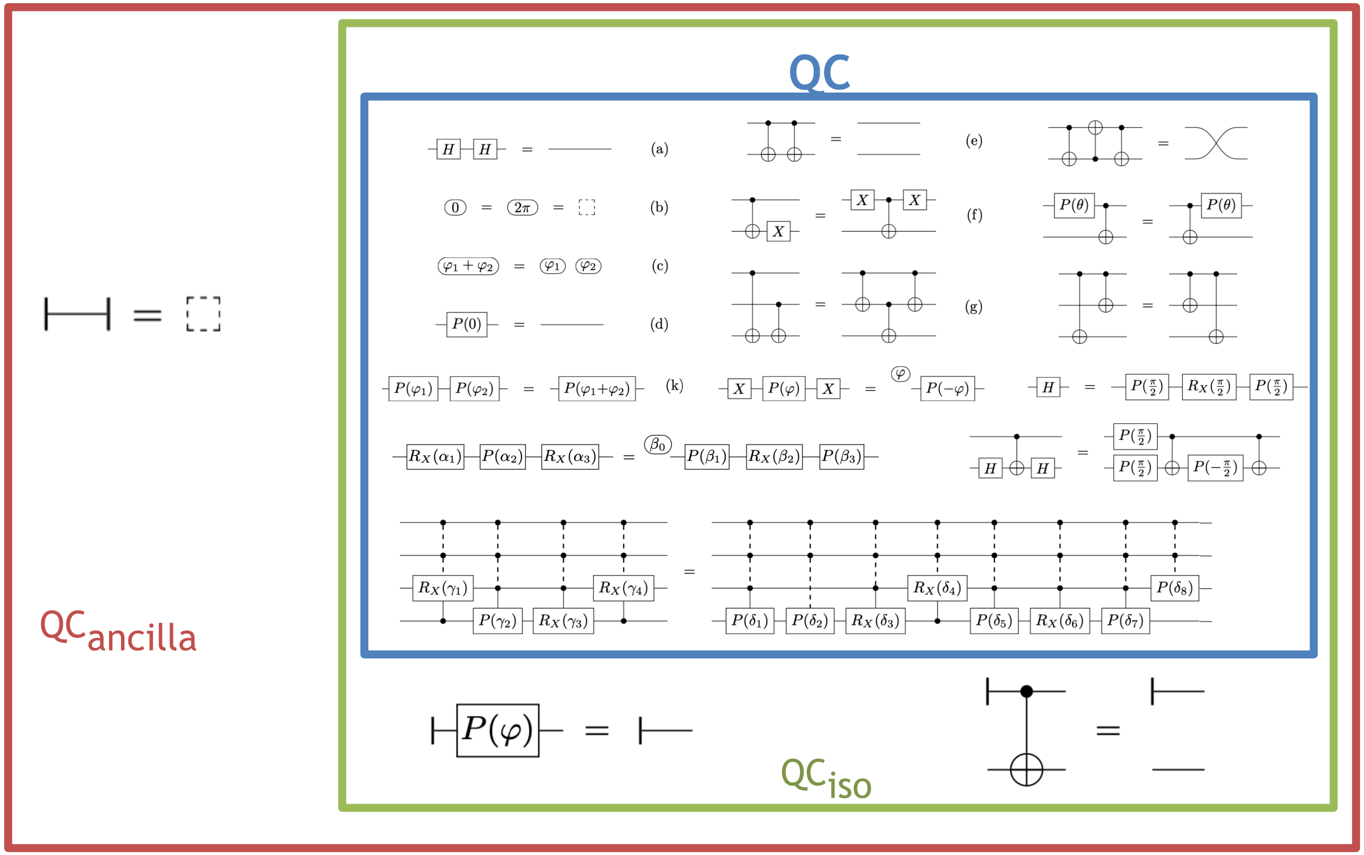
Quantum Circuits with
ancilla and/or trace out



Lemma. QC$_{iso}$ is complete for quantum circuits with qubit initialisation[1]

1 Using CSD or following S. Staton. Algebraic Effects, Linearity, and Quantum Programming Languages. POPL15

Quantum Circuits with
ancilla and/or trace out

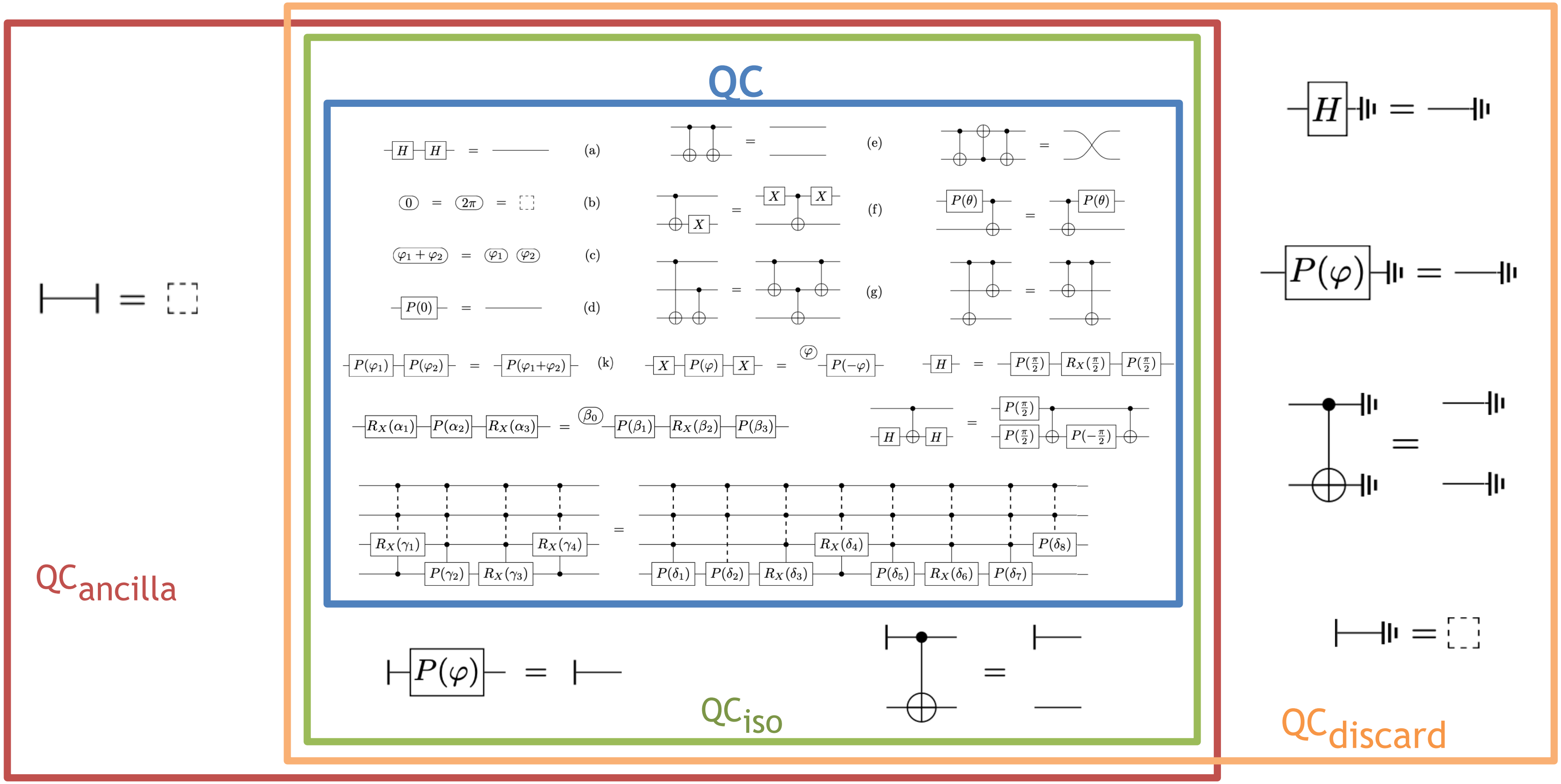


Lemma. QC$_{\text{iso}}$ is complete for quantum circuits with qubit initialisation[1]

Lemma. QC$_{\text{ancilla}}$ is complete for quantum circuits with ancilla

1 Using CSD or following S. Staton. Algebraic Effects, Linearity, and Quantum Programming Languages. POPL15

Quantum Circuits with
ancilla and/or trace out



Lemma. QC$_{\text{iso}}$ is complete for quantum circuits with qubit initialisation[1]

Lemma. QC$_{\text{ancilla}}$ is complete for quantum circuits with ancilla

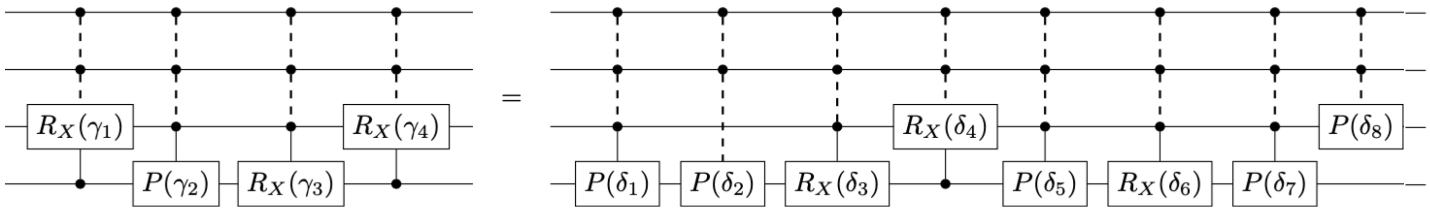
Lemma. QC$_{\text{discard}}$ is complete for quantum circuits with discard[2]

1 Using CSD or following S. Staton. Algebraic Effects, Linearity, and Quantum Programming Languages. POPL15

2 Using construction from M. Huot and S. Staton. Quantum channels as a categorical completion. LICS19 or Carette, Jeandel, Perdrix Vilmart, discard construction ICALP19

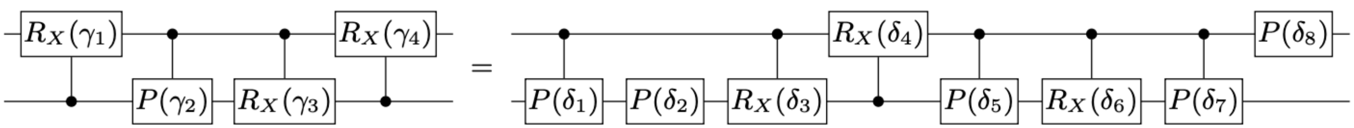
Simplifying (r)

Theorem. In $\mathrm{QC}_{\mathrm{ancilla}}$ and $\mathrm{QC}_{\mathrm{discard}}$, the family of equations

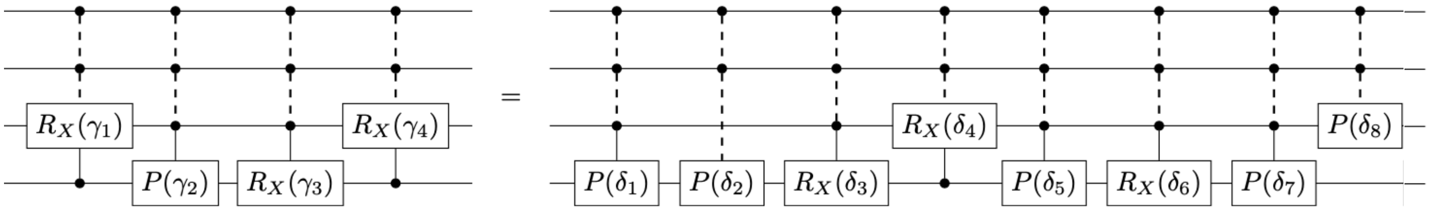


(r)

can be replaced by its 2-qubit case:

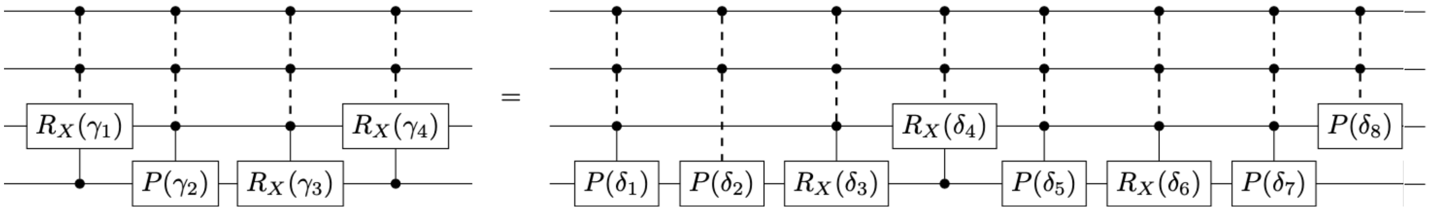


Simplifying (r)

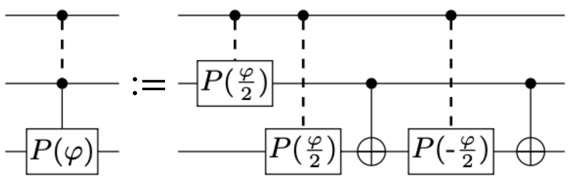


(r)

Simplifying (r)

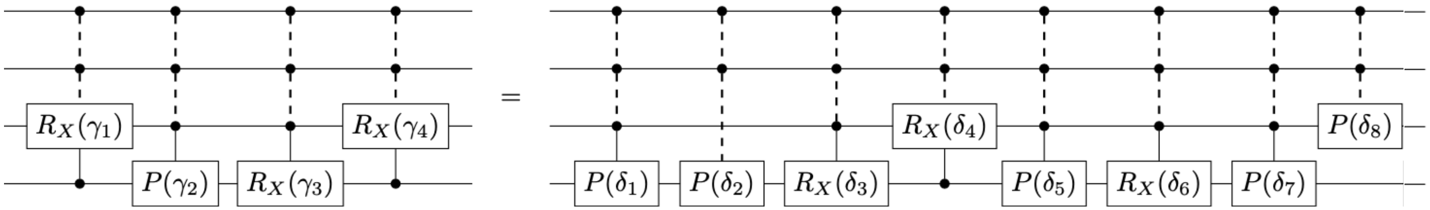


(r)

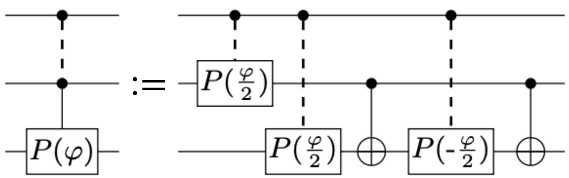


Simplifying (r)

$\forall \gamma_i, \exists \delta_j$ such that



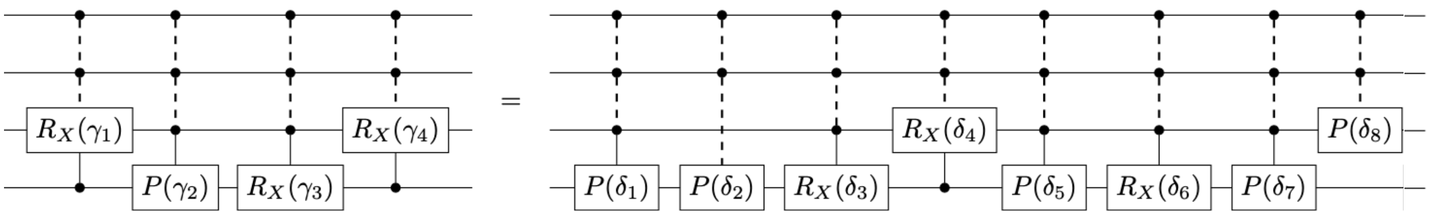
(r)



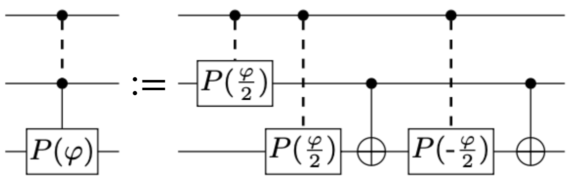
$$
\begin{aligned}
\delta_1 &= f_1(\gamma_1, \ldots, \gamma_4) \\
\delta_2 &= f_2(\gamma_1, \ldots, \gamma_4) \\
&\vdots \\
\delta_9 &= f_9(\gamma_1, \ldots, \gamma_4)
\end{aligned}
$$

Simplifying (r)

$\forall \gamma_i, \exists \delta_j$ such that



(r)



$$
\begin{aligned}
\delta_1 &= f_1(\gamma_1, \ldots, \gamma_4) \\
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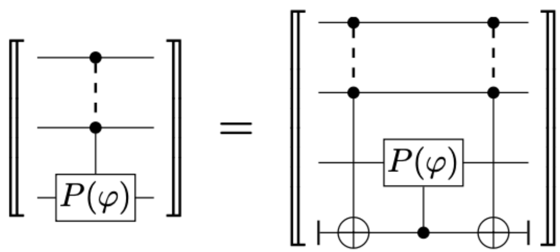
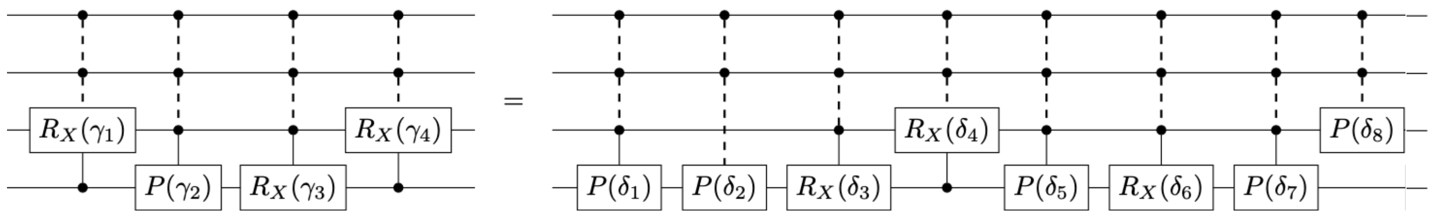
&\vdots \\
\delta_9 &= f_9(\gamma_1, \ldots, \gamma_4)
\end{aligned}
$$

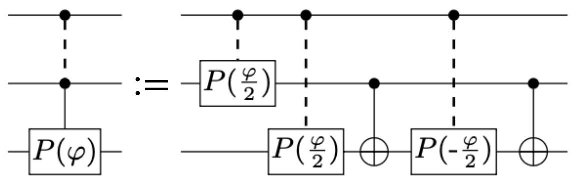


Simplifying (r)

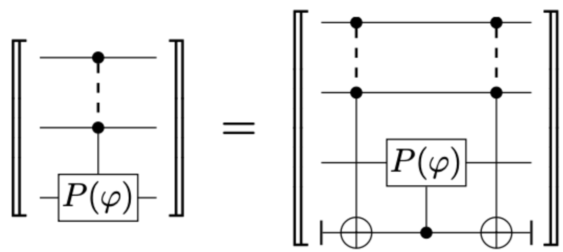
$\forall \gamma_i, \exists \delta_j$ such that



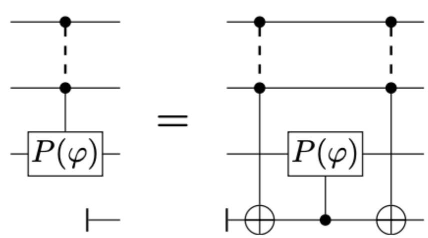
(r)



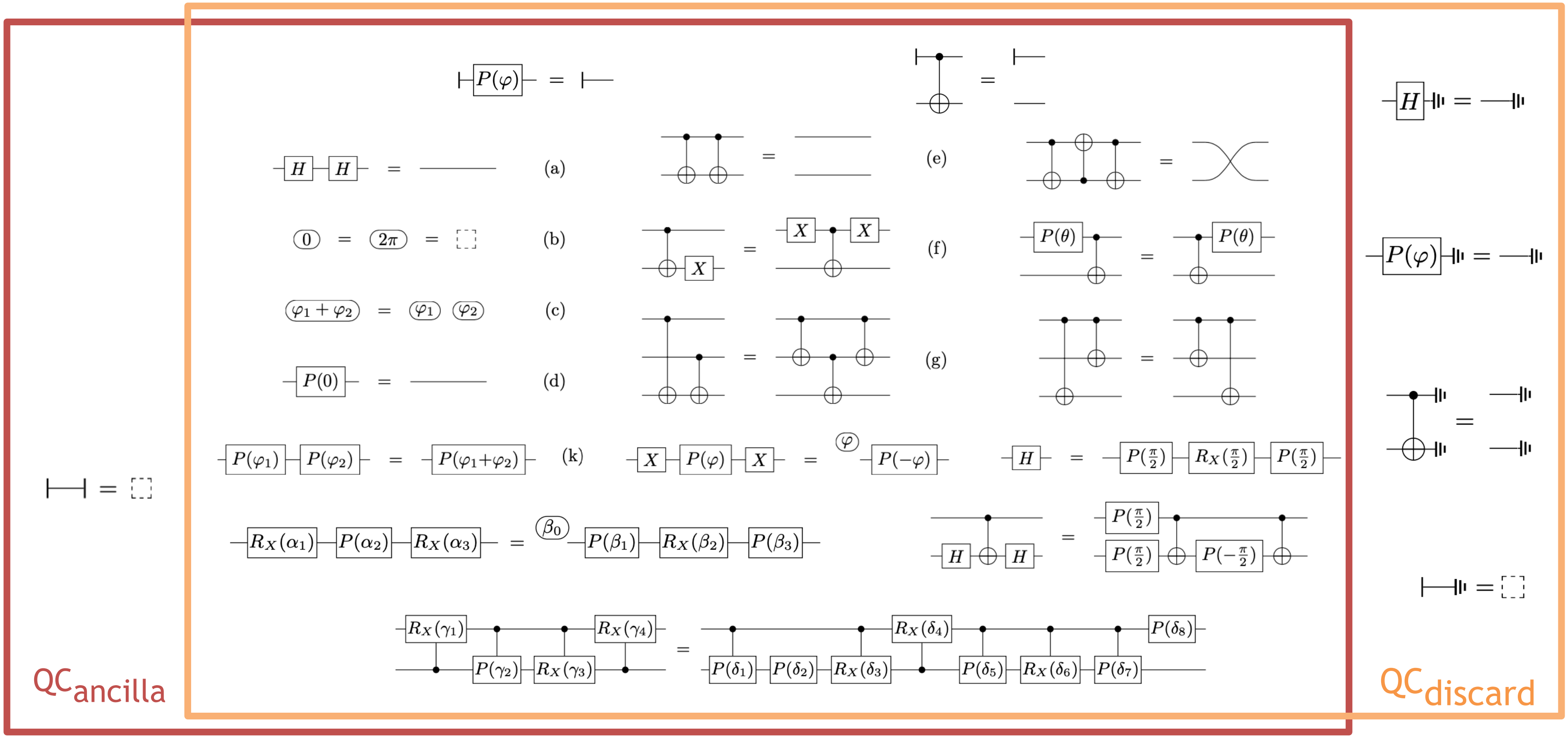
$$\begin{aligned} \delta_1 &= f_1(\gamma_1, \ldots, \gamma_4) \\ \delta_2 &= f_2(\gamma_1, \ldots, \gamma_4) \\ &\vdots \\ \delta_9 &= f_9(\gamma_1, \ldots, \gamma_4) \end{aligned}$$



Lemma. $\mathrm{QC}_{\text{ancilla}} \setminus (r) \vdash$



Quantum Circuits with
ancilla and/or trace out



Concluding remarks

Simplifying two out of the three most complicated rules

Complete equational theories for:

- Quantum circuits with qubit-initialisation
- Quantum circuits with ancilla
- Quantum circuits with initialisation and discard

Complete equational theories acting on at most 3 qubits for QC_{ancilla} and QC_{discard}.

Quantum circuit reasoning in action.