Counting perfect matchings with the planar fragment of ZW-Calculus

Titouan Carette, Etienne Moutot,
Thomas Perez, and Renaud Vilmart

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- **FKT algorithm** : efficiently simulable fragment (like Clifford fragment for the ZX-calculus)
- **In this talk** : alternative diagrammatical polynomial algorithm counting perfect matchings in planar graphs using **pW**-calculus

1. Perfect Matchings
2. ZW-calculus, **pW**-calculus
3. Rewriting strategy
4. A diagrammatical algorithm
5. Further work

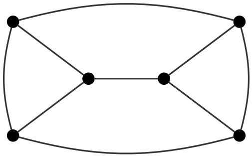
Summary

Perfect matchings

Definition

Let $G = (V, E)$ be a graph.
We say that $P \subset E$ is a perfect matching when every vertex is covered by exactly one edge in P.

Example



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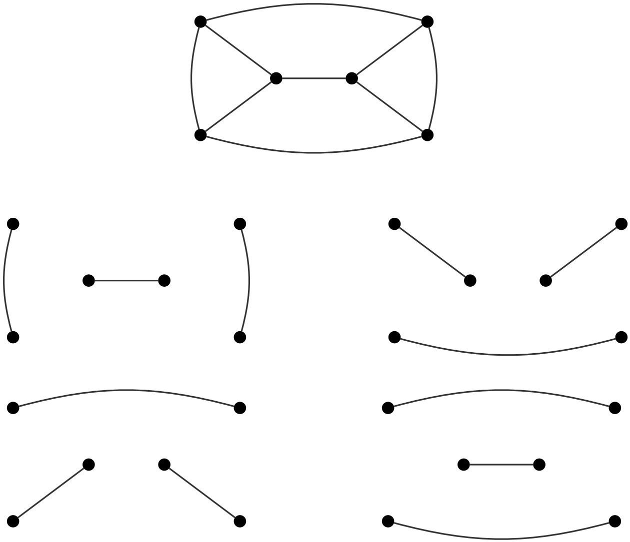

Counting perfect matching

- In general, $\#P$-Complete [Valiant, '79].

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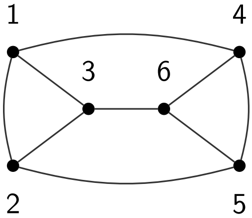
- Polynomial for planar graphs [Fisher-Kasteleyn-Temperley, '67]

FKT algorithm example



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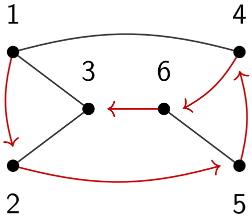

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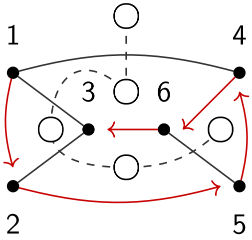

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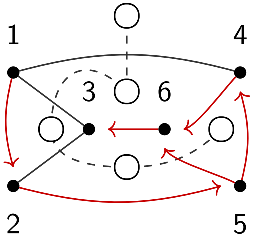

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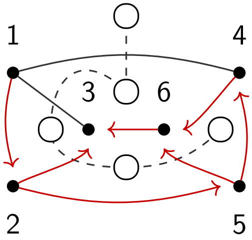

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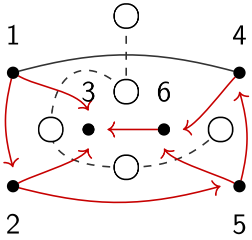

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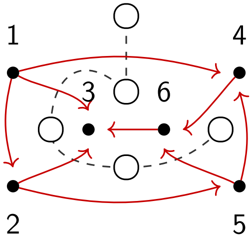

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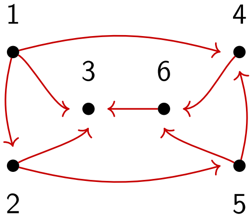

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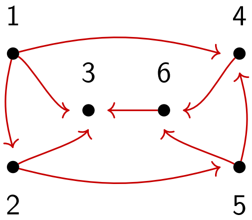

$$\#PM = Pf(M) = \sqrt{det(M)} = 4$$

Summary

ZW-calculus

- Bob Coecke and Aleks Kissinger : The compositional structure of multipartite quantum entanglement.

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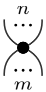
- Bob Coecke and Aleks Kissinger : The compositional structure of multipartite quantum entanglement.
- Amar Hadzihasanovic : A diagrammatic axiomatisation for qubit entanglement.
- Interesting connections with fermionic quantum computing, and recent works importing some ZW-calculus primitives into ZX-calculus

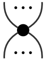
Planar W-calculus



$$:= \sum_{\substack{u\in\{0,1\}^m\\ v\in\{0,1\}^n\\ |uv|=1}} |u\rangle\langle v|$$

$$:= |0\rangle\langle 0| + r\,|1\rangle\langle 1|$$

Planar W-calculus



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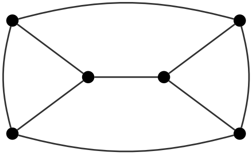
Correspondence

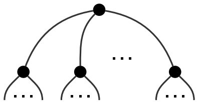
The interpretation of the **pW**-scalar associated to a graph is its number of perfect matchings.

Example



$= 4$

Vertex by vertex Decomposition



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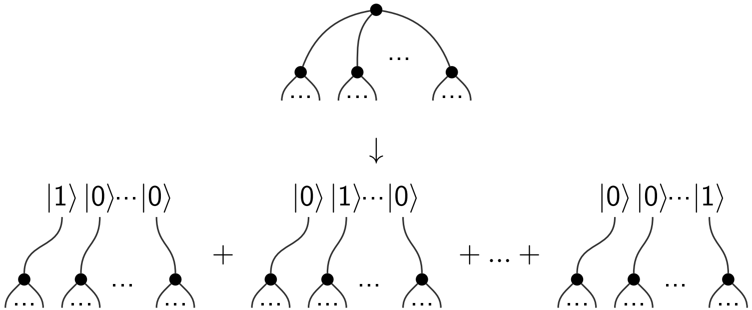

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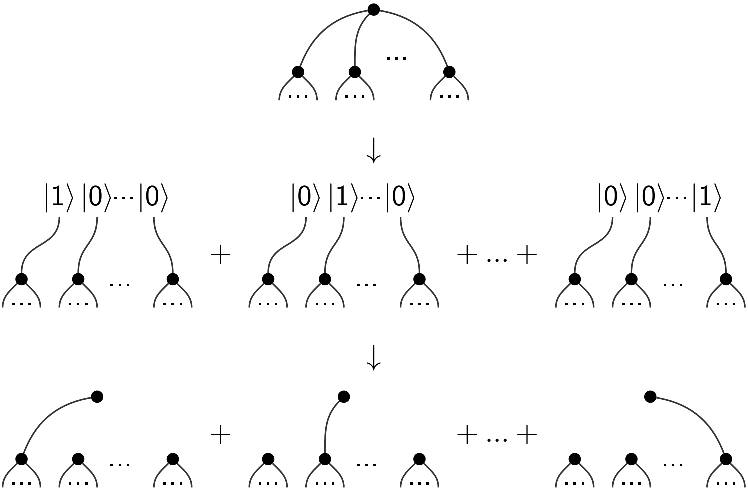

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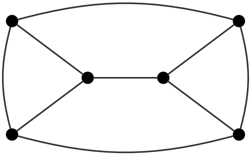

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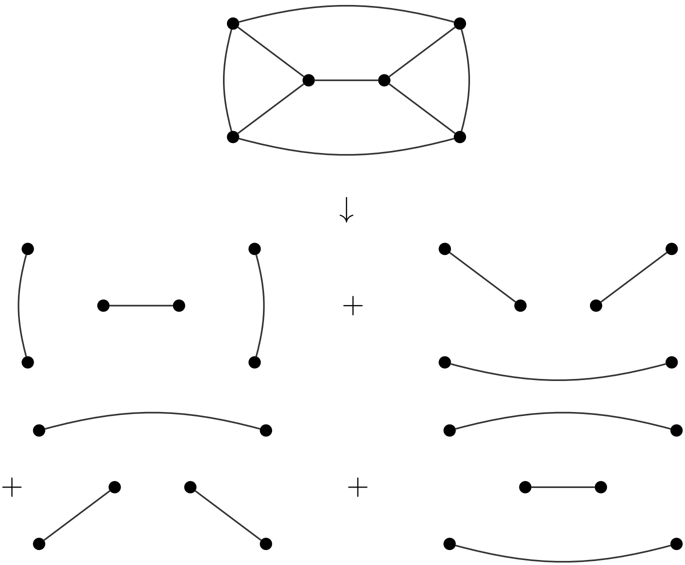
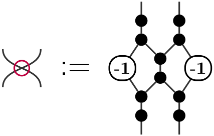


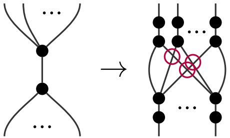
Fermionic Swap

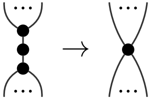


$$:= \sum_{x,y\in\{0,1\}} (-1)^{xy} \, |xy\rangle\langle yx|$$

Summary

Rewriting Strategy





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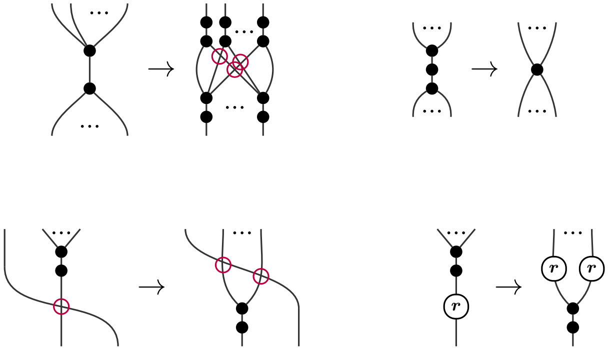

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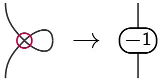

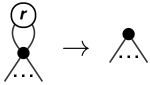

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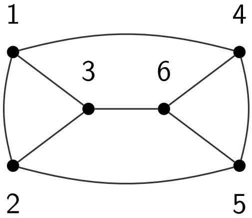

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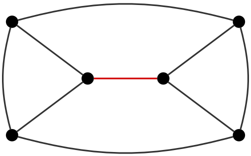

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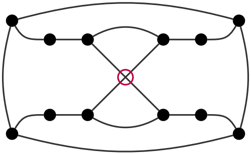

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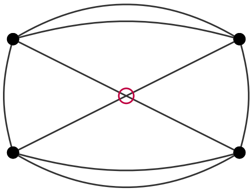

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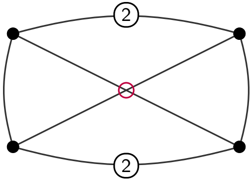

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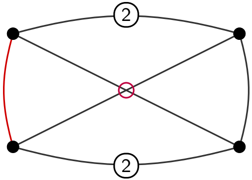

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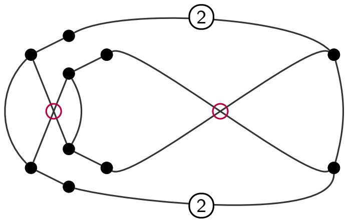

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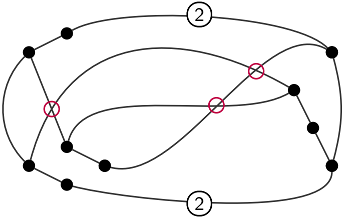

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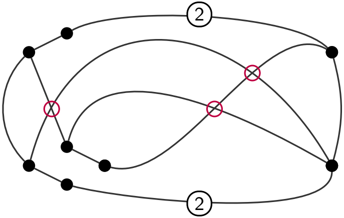

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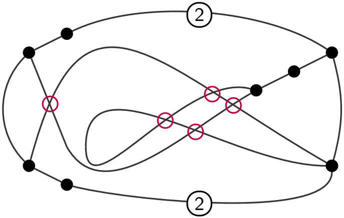

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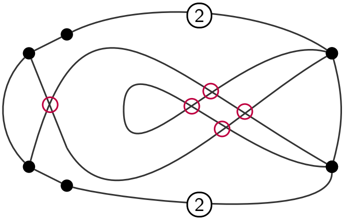

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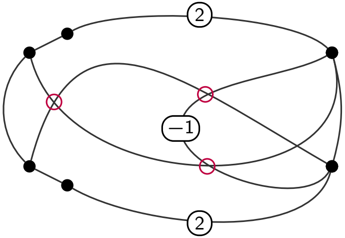

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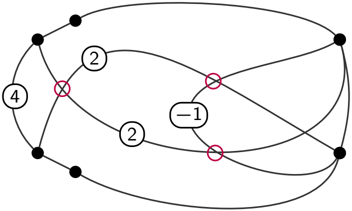

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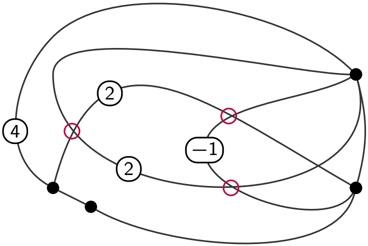

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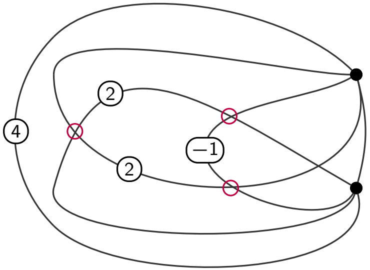

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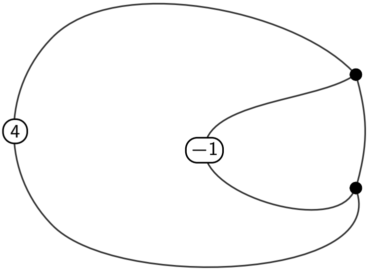

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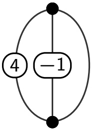

Example

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Embedding

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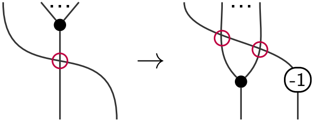
- **Solution** : Let's embed the diagram associated to our graph on a semi-circle.
- **Advantage** : One can consider the nodes ordered from left to right, use this order to know on which edge to apply the rewriting strategy, and deduce whether or not two wires intersect each other via a fermionic swap from the order on their vertices.

Embedding a diagram



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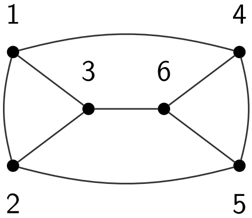

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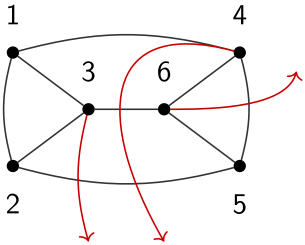

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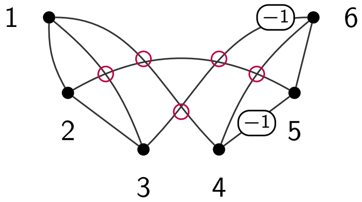

- **Question** : How to embed our diagram in practice ?

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- **Solution** : We don't need to do it graphically, and can use a Pfaffian orientation to put -1 weights on the right edges.

Proposition

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The interpretation of a diagram associated to a graph embedded on a semi-circle is the Pfaffian of its adjacency matrix.

Pfaffian

Definition

Let $M \in \mathbb{C}^{2n \times 2n}$,

$$Pf(M) := \sum_{\substack{\pi \in S_{2n} s.t. \forall i \\ \pi(2i-1) < \pi(2i) \\ \pi(2i-1) < \pi(2i+1)}} \epsilon(\pi) \prod_{i=1}^{n} M_{\pi(2i-1),\pi(2i)}$$

Sketch of proof

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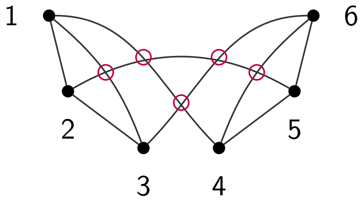
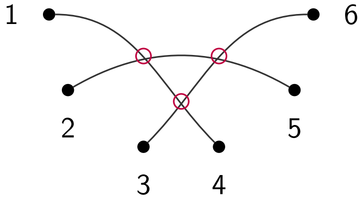


Sketch of proof

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 3 & 6 \end{pmatrix} : \qquad \epsilon(\pi) \prod_{i=1}^{n} M_{\pi(2i-1),\pi(2i)}$$



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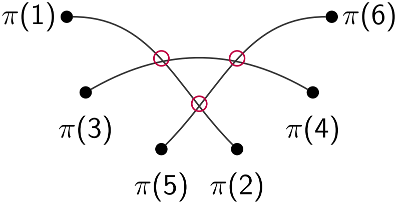

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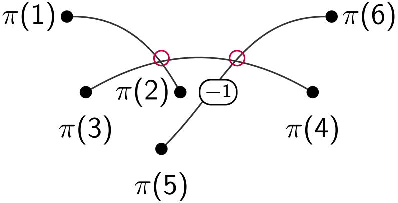

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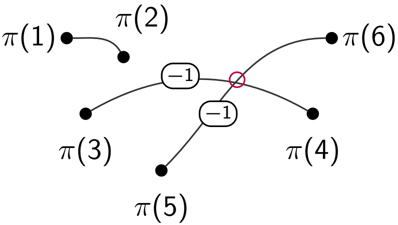

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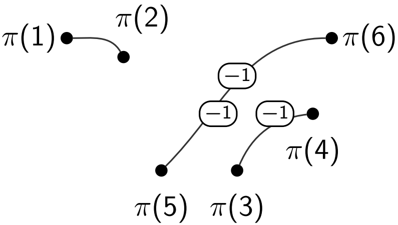

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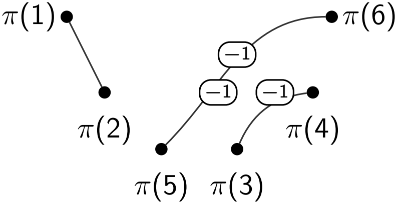

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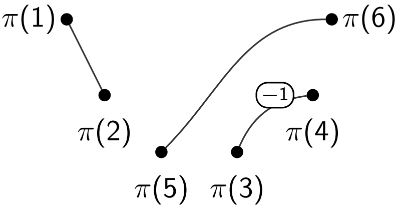

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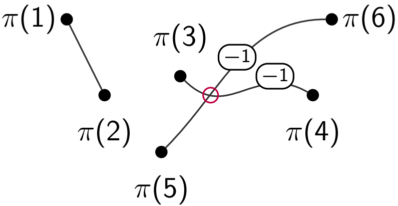

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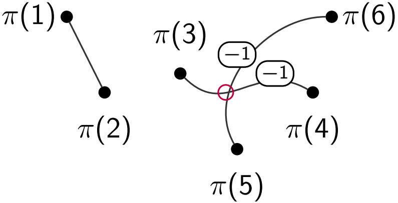

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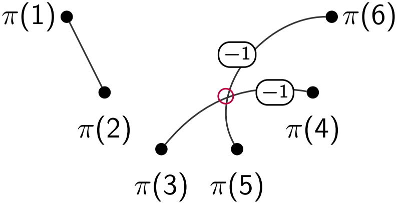

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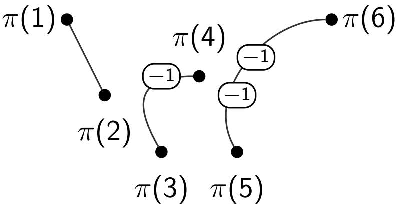

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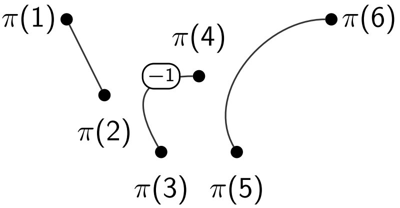

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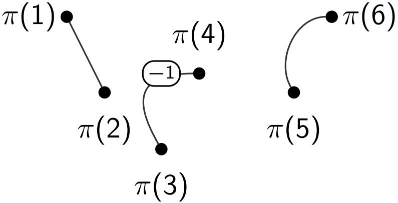

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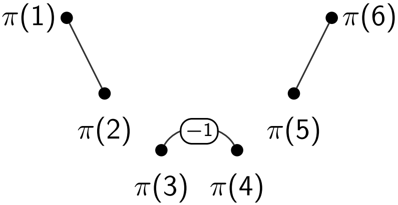

Algorithm

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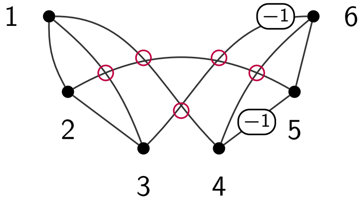
3. While there is more than 2 black spiders, apply the rewriting strategy on edge (i, j), i being minimal and j maximal

Example



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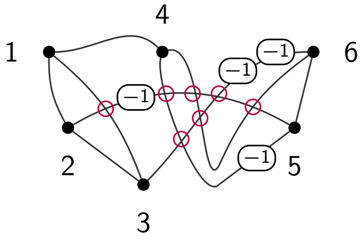

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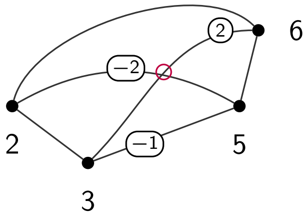

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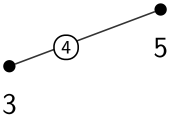

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- Gives a graphical interpretation of the role of the Pfaffian oriantation.

Summary

Further work

- Adapt the algorithm to compute scalars in the fermionic ZW-calculus (**pW** $+$ swap gate) which is universal for **Qubits** modulo an encoding trick.

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Further work

- Adapt the algorithm to compute scalars in the fermionic ZW-calculus (**pW** + swap gate) which is universal for **Qubits** modulo an encoding trick.
- Extend the polynomial algorithm to all graphs for which counting perfect matchings is polynomial.
- Explore the link between Pfaffian orientations and planar embeddings of a graph.

The end

Thank you for your attention