The Qupit Stabiliser ZX-travaganza:

Simplified Axioms, Normal Forms and Graph-Theoretic Simplification

arXiv:2306.05204

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QPL 2023

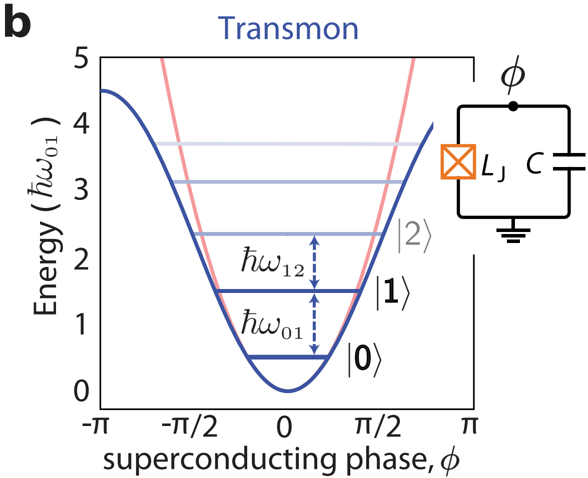
Qubits:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

Qudits:

$$|\psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle + a_2\,|2\rangle + \cdots + a_{d-1}\,|d-1\rangle$$

Physical Realisation of Qudits



Kjaergaard et al., 2020

Qupits are the odd prime dimensional qudits

Complete ZX-Calculi for the Stabiliser Fragment in Odd Prime Dimensions

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Abstract

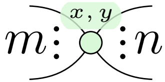
We introduce a family of ZX-calculi which axiomatise the stabiliser fragment of quantum theory in odd prime dimensions. These calculi recover many of the nice features of the qubit ZX-calculus which were lost in previous proposals for higher-dimensional systems. We then prove that these calculi are complete, i.e. provide a set of rewrite rules which can be used to prove any equality of stabiliser quantum operations. Adding a discard construction, we obtain a calculus complete for

Booth and Carette, 2022

New results and improvements

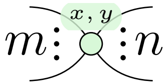
- Reduced axioms
- Well-tempered
- Minimality
- Graph theoretic simplifications
    - Local complementation
    - Pivoting
- Normal forms
- Neat completeness
- Clifford simplification

Generator: Z spider



$$\left[\!\left[\;\cdot\; \right]\!\right] = p^{\frac{n+m-2}{4}} \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} |k : Z\rangle^{\otimes n} \langle k : Z|^{\otimes m}$$

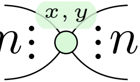
Generator: Z spider



$$[\![\, \cdot \,]\!] = p^{\frac{n+m-2}{4}} \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} |k : Z\rangle^{\otimes n} \langle k : Z|^{\otimes m}$$

Cui, Gottesman and Krishna, 2017

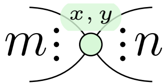
Generator: Z spider



$$\left[\!\left[\, m \vdots \cdots \vdots n \,\right]\!\right] = p^{\frac{n+m-2}{4}} \sum_{k\in\mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} |k:Z\rangle^{\otimes n} \langle k:Z|^{\otimes m}$$

Cui, Gottesman and Krishna, 2017
de Beaudrap, 2021

Generator: Z spider



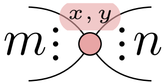
$$\left[\!\!\left[\; \cdot \; \right]\!\!\right] = p^{\frac{n+m-2}{4}} \sum_{k\in\mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} \left|k : Z\right\rangle^{\otimes n} \left\langle k : Z\right|^{\otimes m}$$

Cui, Gottesman and Krishna, 2017

de Beaudrap, 2021

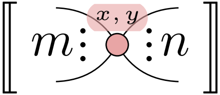
$|0\rangle, |1\rangle, \cdots$

Generator: X spider



$$\left\llbracket \; \cdot \; \right\rrbracket = p^{\frac{n+m-2}{4}} \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} \left| -k : X \right\rangle^{\otimes n} \left\langle k : X \right|^{\otimes m}$$

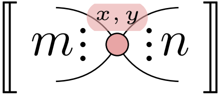
Generator: X spider



$$\left[\!\left[\text{(spider } m \text{ inputs}, n \text{ outputs}, x, y) \right]\!\right] = p^{\frac{n+m-2}{4}} \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} \left|-k : X\right\rangle^{\otimes n} \left\langle k : X\right|^{\otimes m}$$

like $|+\rangle$, $|-\rangle$

Generator: X spider

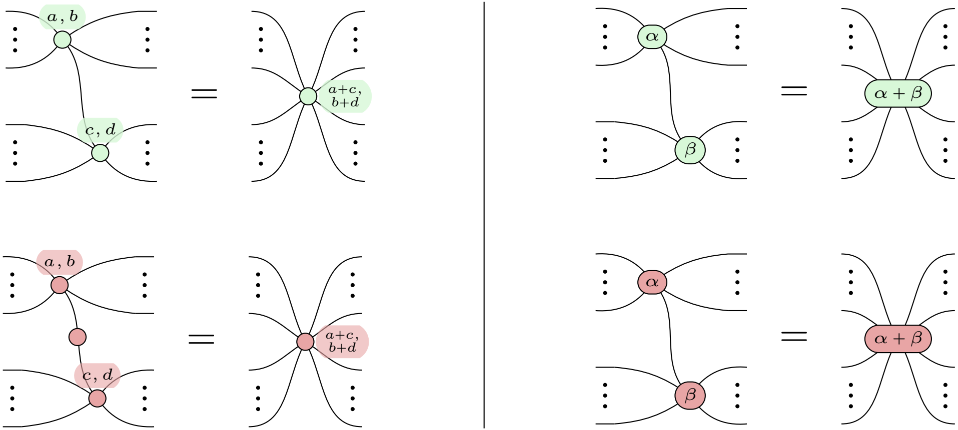


$$\left[\!\!\left[\; m \vdots \overset{x,y}{\bullet} \vdots n \;\right]\!\!\right] = p^{\frac{n+m-2}{4}} \sum_{k\in\mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} \left|-k : X\right\rangle^{\otimes n} \left\langle k : X\right|^{\otimes m}$$

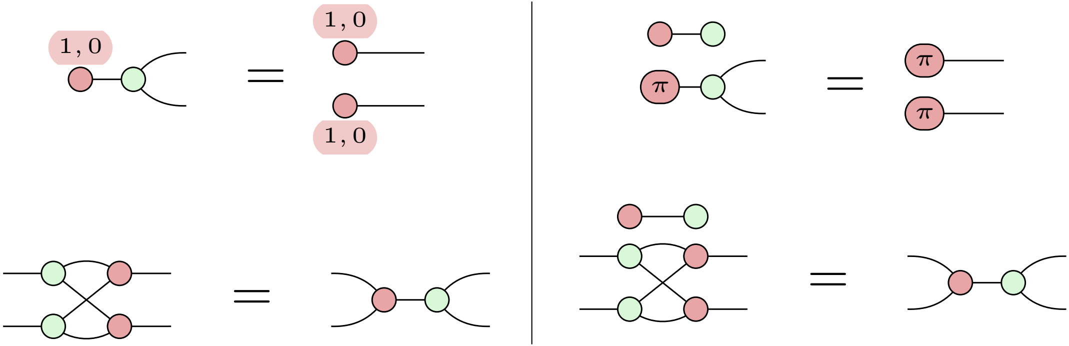
like $|+\rangle$, $|-\rangle$

Carette, 2021

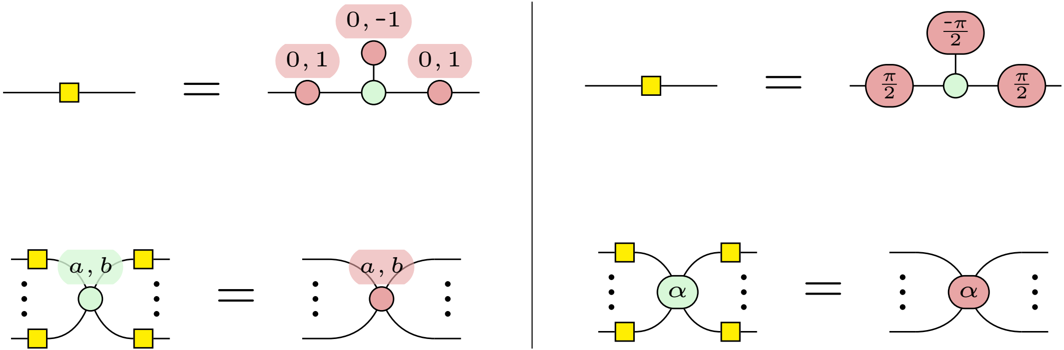
Fusion



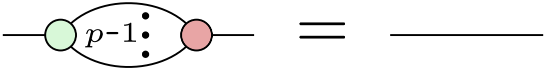
Copy, Bialgebra

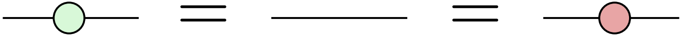


Euler, Colour

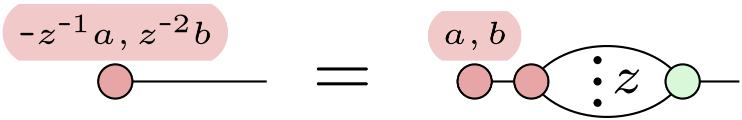


Special / ID





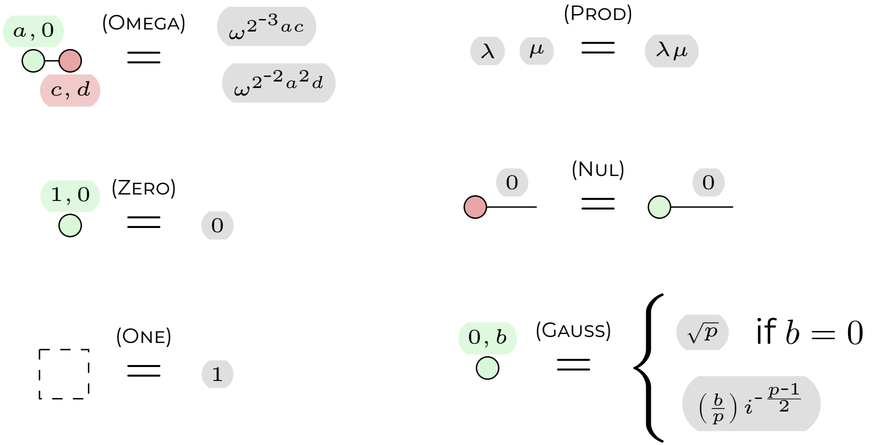
Multiplier Elimination

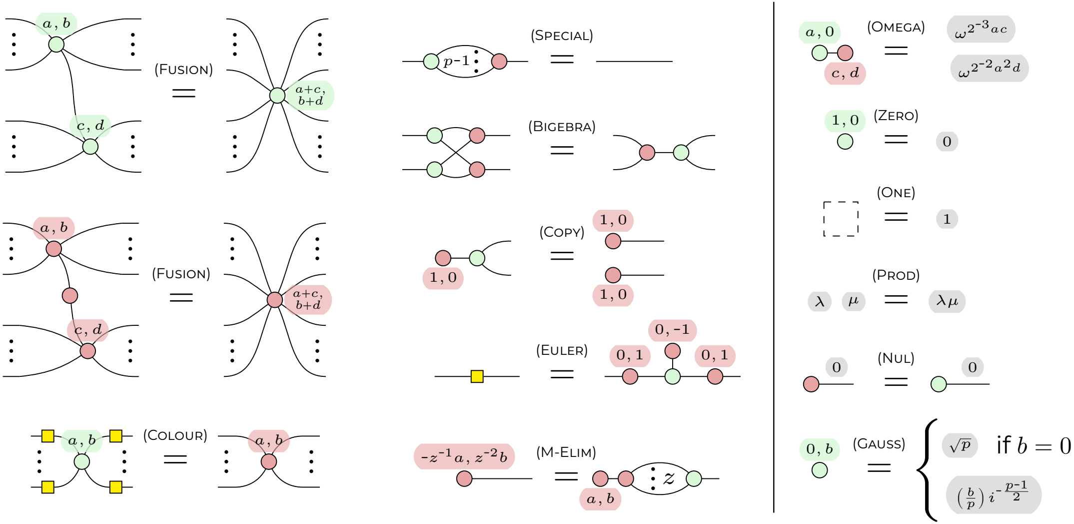


Generator: Explicit scalar

$$[\![s]\!] = \boldsymbol{s}$$

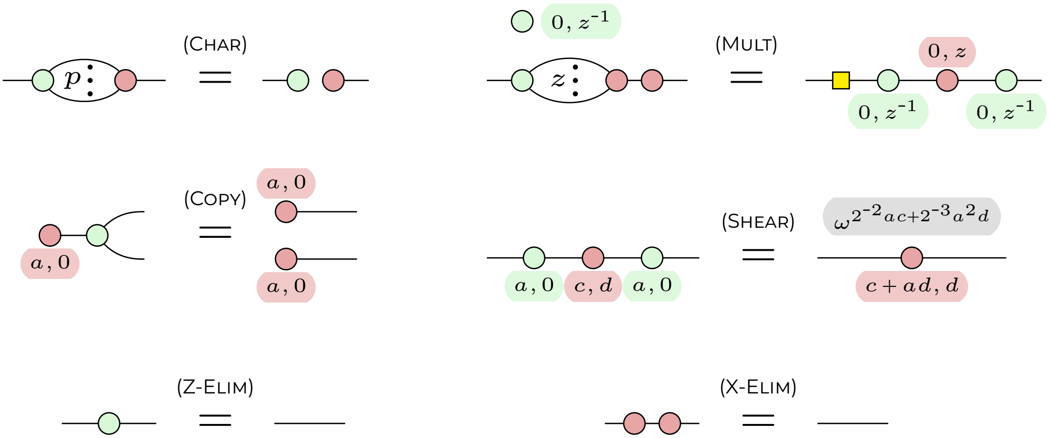
Scalar axioms

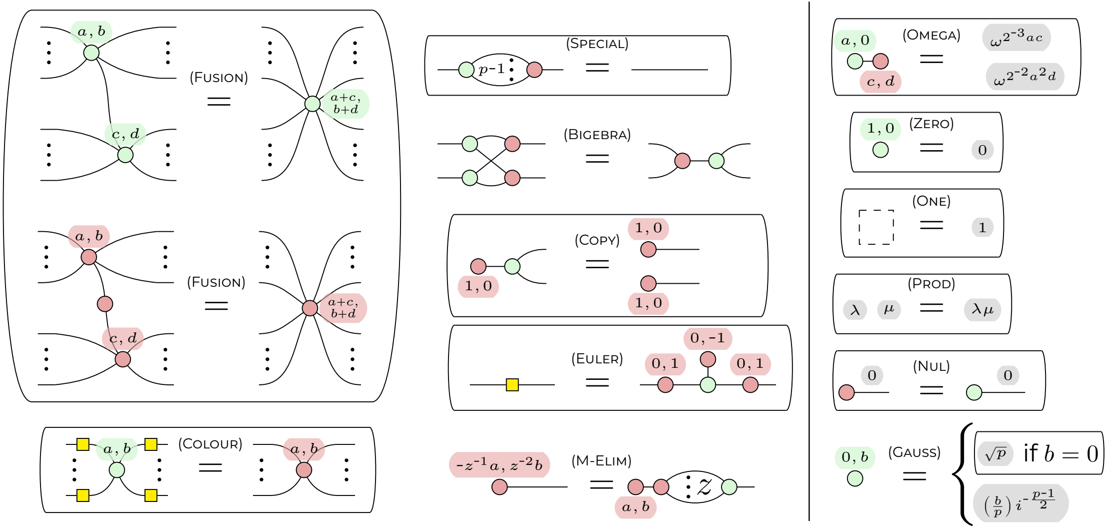




(FUSION)
(FUSION)
(COLOUR)
(SPECIAL)
(BIGEBRA)
(COPY)
(EULER)
(M-ELIM)
(OMEGA)
(ZERO)
(ONE)
(PROD)
(NUL)
(GAUSS)
if $b = 0$

Derived axioms of (Booth and Carette, 2022)





(Fusion)
(Fusion)
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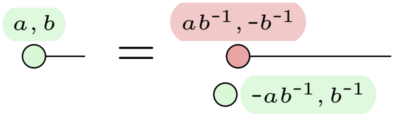
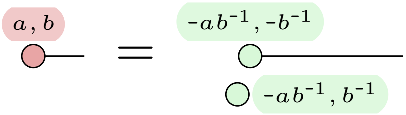
(Special)
(Bigebra)
(Copy)
(Euler)
(M-Elim)
(Omega)
(Zero)
(One)
(Prod)
(Nul)
(Gauss)
if $b = 0$

Strictly-Clifford basis change

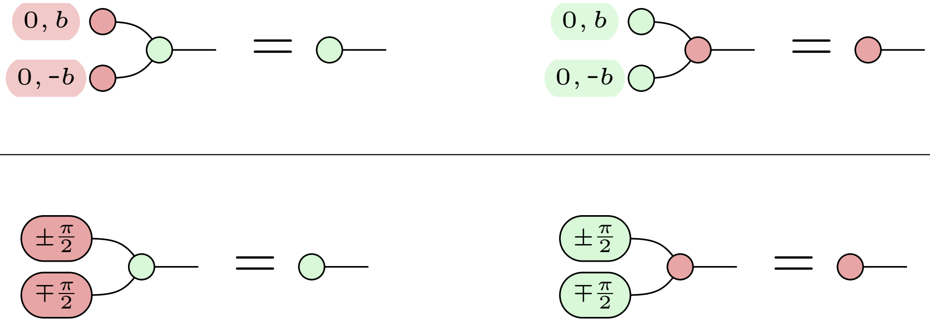
For $a \in \mathbb{Z}_p$ and $b \in \mathbb{Z}_p^*$,





Supplementarity

For $b \in \mathbb{Z}_p^*$,

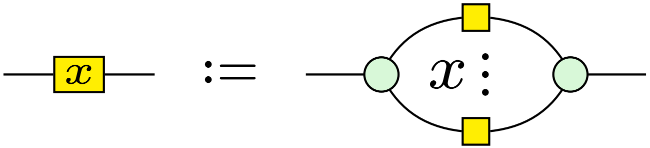


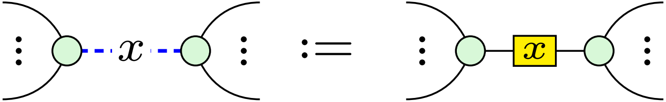
Perdrix and Wang, 2016

Graph-Theoretic Simplifications

Duncan, Kissinger, Perdrix and Wetering, 2020

H-box

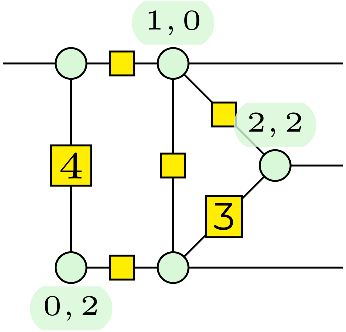




Graph-Like Diagrams

A ZX-diagram is graph-like when it has

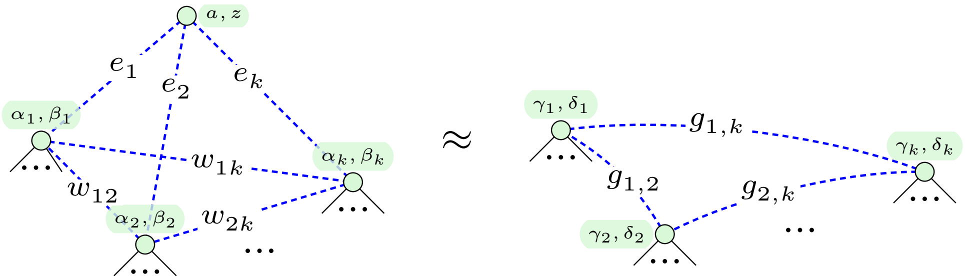
1. only Z-spiders,
2. only H-edges,
3. no self-loops,
4. all its inputs and outputs connected to a Z-spider,
5. no Z-spider connected to more than one input or output.



Any ZX-diagram can be transformed into a graph-like diagram

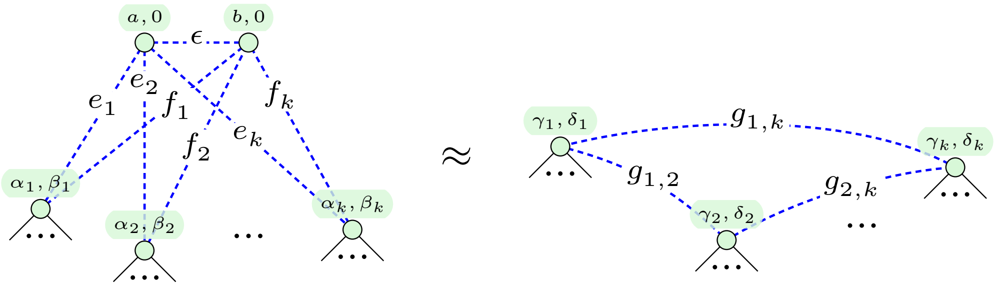
Local Complementation

For $* \in \mathbb{Z}_p$ and $z \in \mathbb{Z}_p^*$:



Pivoting

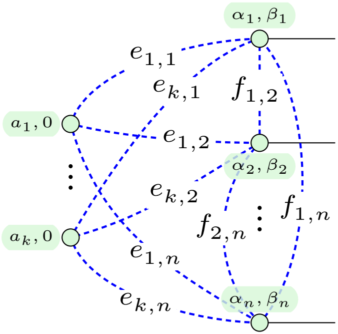
For $* \in \mathbb{Z}_p$ and $\epsilon \in \mathbb{Z}_p^*$:



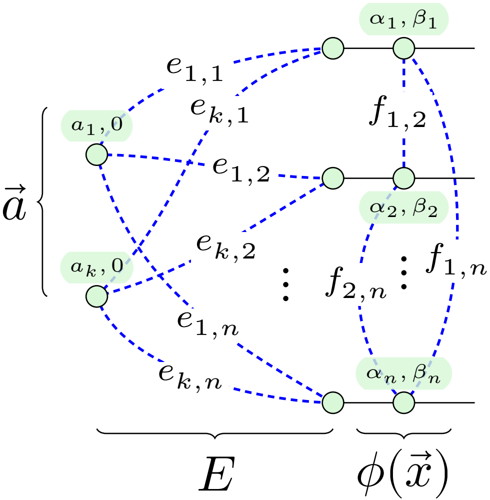
AP-form

A graph-like diagram is in Affine with Phases form (AP-form) when:

- There are no inputs;
- All internal spiders are Pauli;
- Internal spiders not connected to each other.

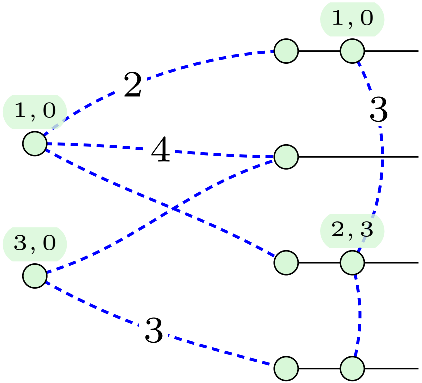


AP-form



$$\overset{[\![\cdot]\!]}{\longmapsto} \quad \sum_{E\vec{x}=\vec{a}} \omega^{\phi(\vec{x})} |\vec{x}\rangle$$

Example



$$\vec{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$E = \begin{pmatrix} 2 & 4 & 1 & 0 \\ 0 & 1 & 0 & 3 \end{pmatrix}$$

$$\phi(\vec{x}) = 2^{-3}x_1 + x_3 + x_3^2 - x_1x_3 - 2^{-3}x_3x_4$$

Reduced AP-form

A diagram in AP-form defined by $(E, \vec{a}, \phi)$ is in reduced AP-form if:

- E is in reduced row echelon form (RREF);
- E contains no fully zero rows.
- ϕ only contains free variables from the equation system of E;

The reduced AP-form is unique

Any diagram can be transformed into reduced AP-form

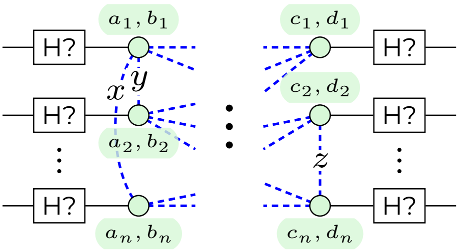
Corollary

Completeness!

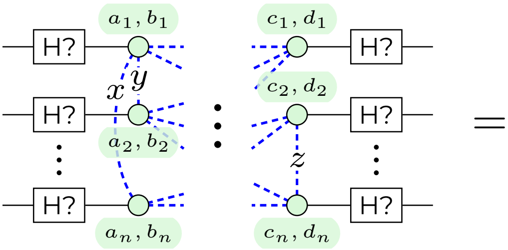
GS-LC form

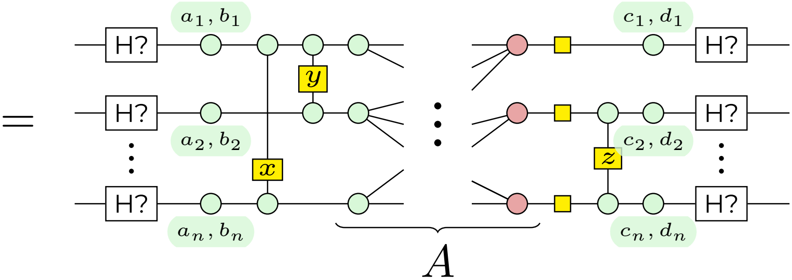
A diagram is in Graph State with Local Cliffords form (GS-LC form) when:

- It is graph-like up to Hadamards on output wires;
- It has no internal spiders.



Decomposing arbitrary Clifford unitaries

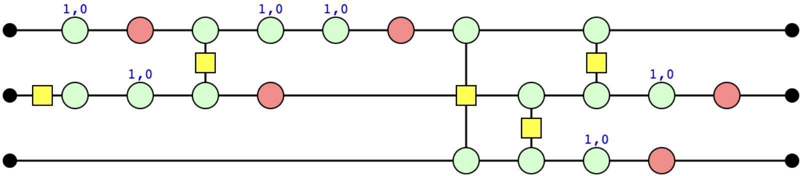




DiZX

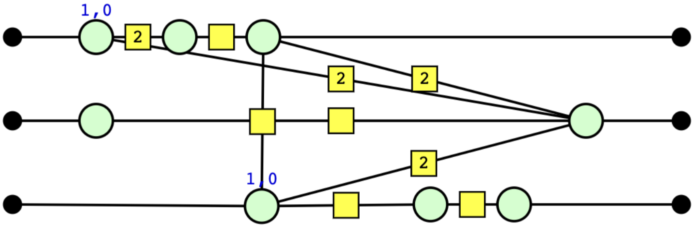
DiZX

```
zcircuit.add_gate("CZ", 2, 1)
zcircuit.add_gate("CZ", 0, 1)
zcircuit.add_gate("Z", 1)
zcircuit.add_gate("Z", 2)
zcircuit.add_gate("NEG", 2)
zcircuit.add_gate("NEG", 1)
g = zcircuit.to_graph(zh=True)
dizx.draw(g)
```



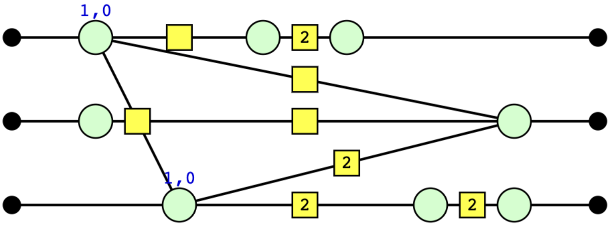
DiZX: AP-from

```
to_ap_form(g)
dizx.draw(g)
```



DiZX: Clifford simp

```
clifford_simp(g)
dizx.draw(g)
```



Summary

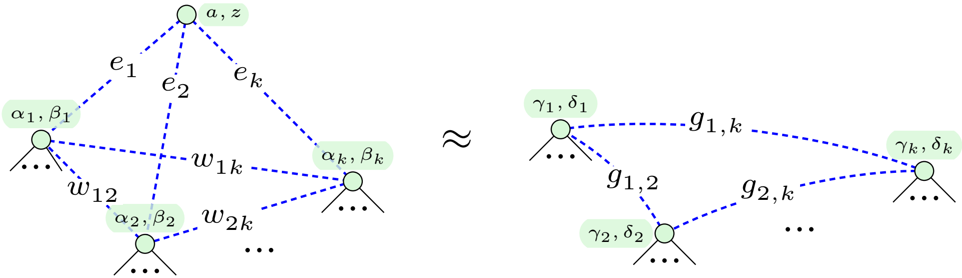
- The calculus is
  - simplified,
  - well-tempered,
  - almost minimal,
  - and has explicit scalars.
- Using it, we
  - show local complementation and pivoting,
  - simplify Clifford diagrams,
  - prove completeness (reduced AP-form),
  - and decompose Clifford unitaries (GS-LC form).
- Try it out in `DiZX`!

Appendix

Overview

Local Complementation

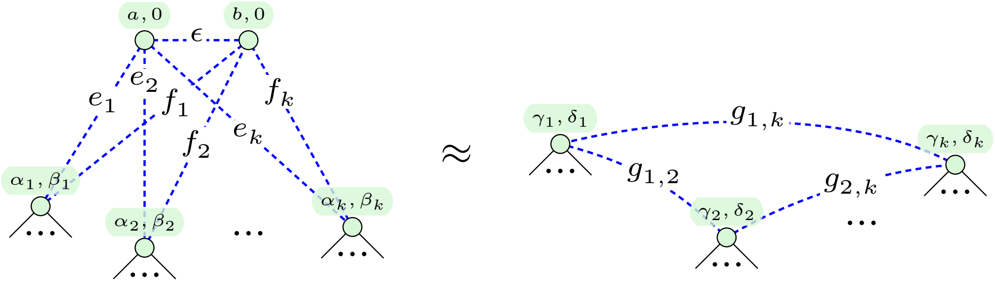
For any $z \in \mathbb{Z}_p^*$ and for all $a, \alpha_i, \beta_i, e_i, w_{i,j} \in \mathbb{Z}_p$ where $i, j \in \{1, \dots k\}$ such that $i < j$ we have:



Here $\gamma_i = \alpha_i - e_i a z^{-1}$, $\delta_i = \beta_i - z^{-1} e_i^2$, and $g_{i,j} = w_{ij} - z^{-1} e_i e_j$.

Pivoting

Here again $\epsilon \in \mathbb{Z}_p^*$ with every other variable on the left-hand side allowed arbitrary values.



On the right-hand side $\gamma_i = \alpha_i - \epsilon^{-1}(af_i + be_i)$, $\delta_i = \beta_i - 2\epsilon^{-1}e_i f_i$, and $g_{i,j} = -\epsilon^{-1}(e_i f_j + e_j f_i)$.

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