Completeness for arbitrary finite dimensions of ZXW-calculus

arXiv:2302.12135

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QPL 2023

QUANTINUUM

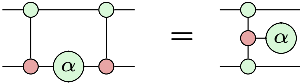
UNIVERSITY OF OXFORD

Prelimenaries

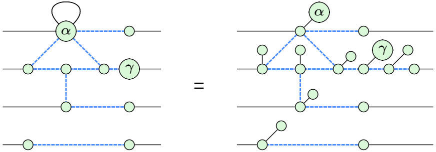
- ZXW-calculus
- Qudits
- Completeness

ZX-calculus

Quantum Circuit Optimisation

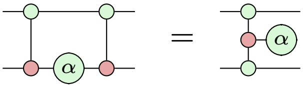


Measurement-Based Quantum Computing

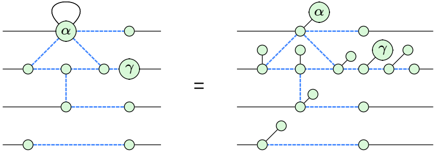


ZX-calculus

Quantum Circuit Optimisation

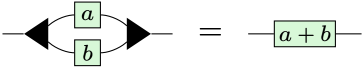


Measurement-Based Quantum Computing

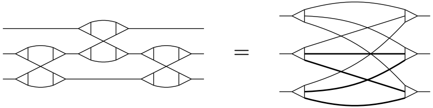


ZW-calculus

Summation

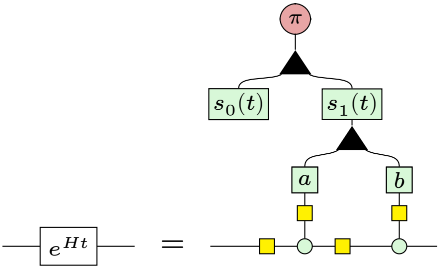


Linear Optical Quantum Computing

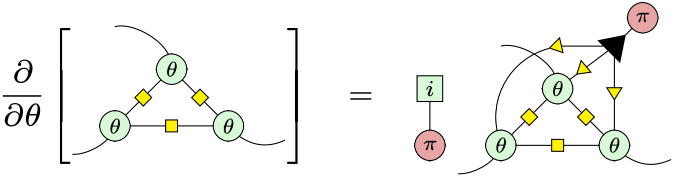


ZXW-calculus

Hamiltonians (Shaikh, Wang and Yeung, 2022)



Differentiation and integration (Wang, Yeung and Koch, 2022)



What are Qudits?

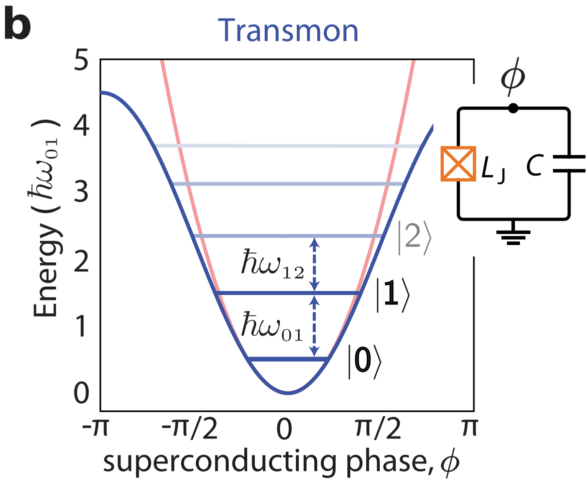
Qubits:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

Qudits:

$$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle + a_2 |2\rangle + \cdots + a_{d-1} |d-1\rangle$$

Physical Realisation of Qudits

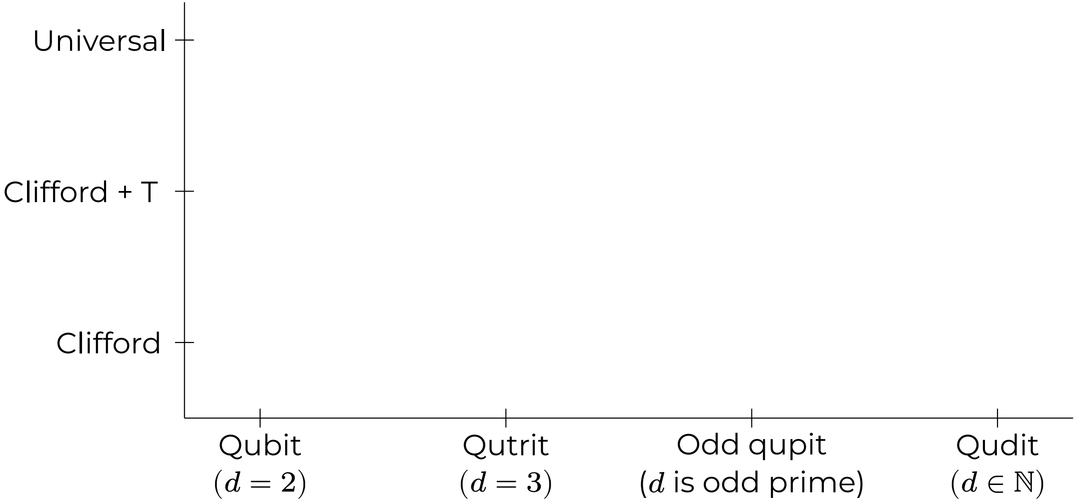


Kjaergaard et al., 2020

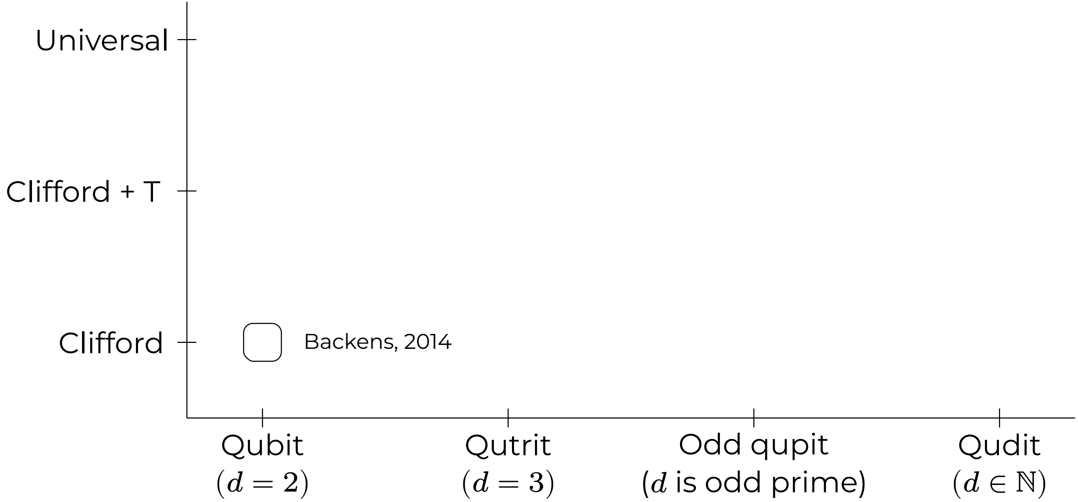
Completeness

If diagrams D_1 and D_2 have the same interpretation, we can prove $D_1 = D_2$ using the rules of the calculus.

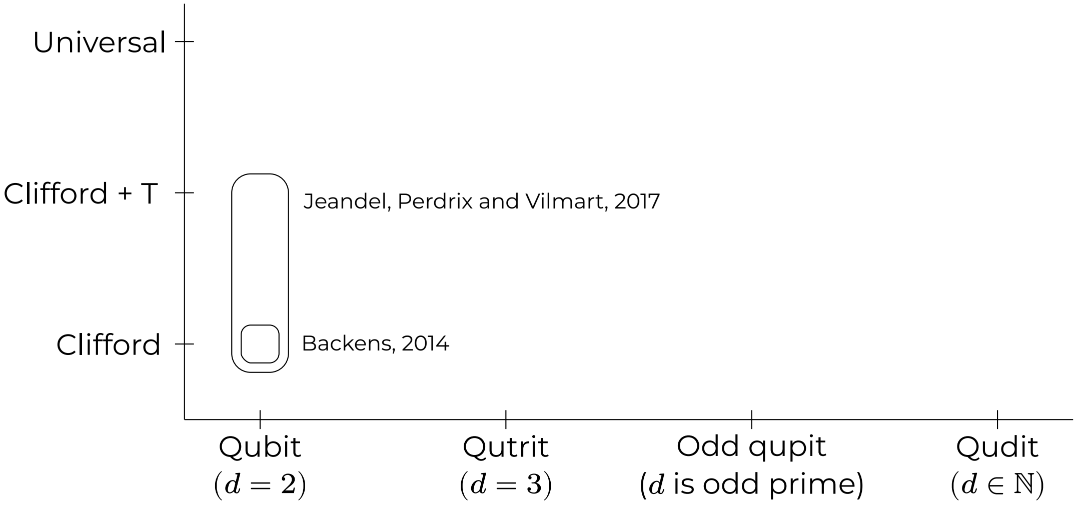
History of Completeness



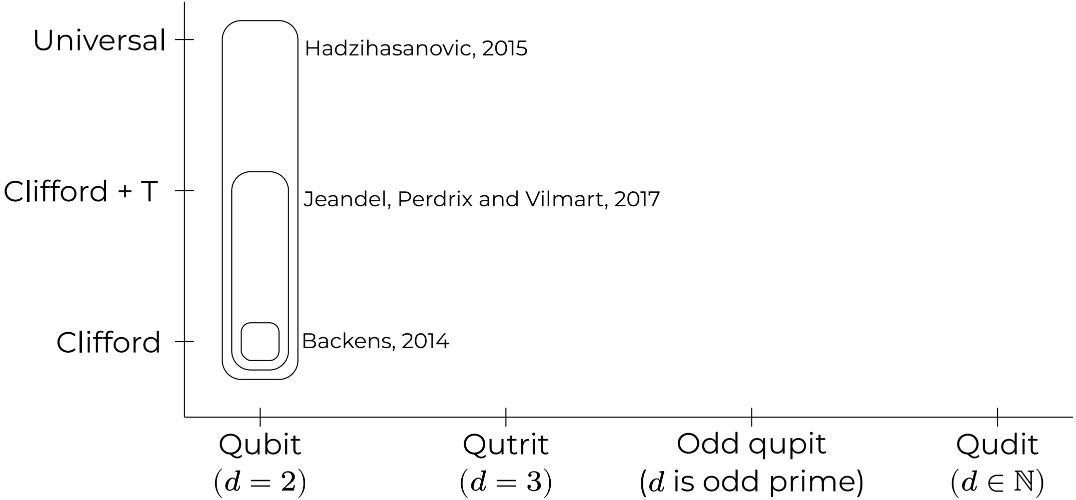
History of Completeness



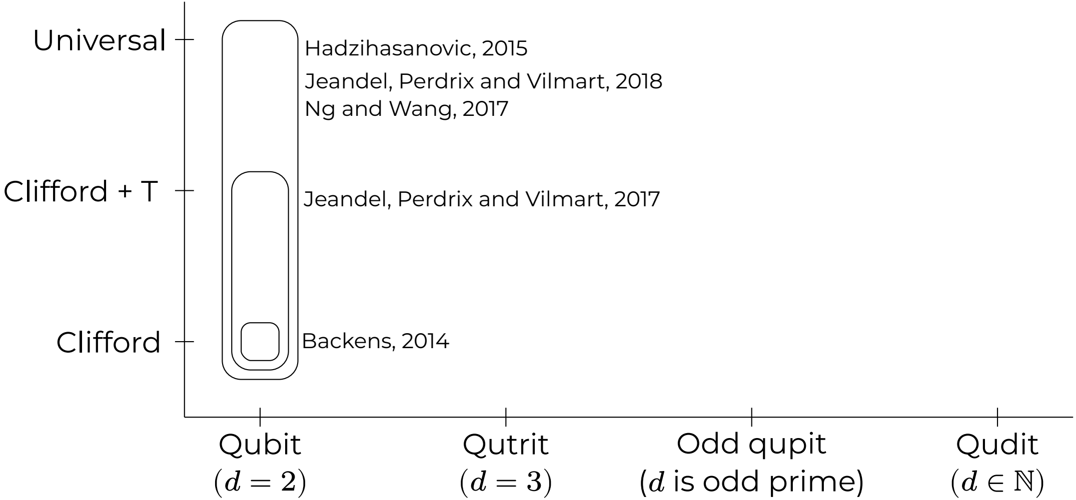
History of Completeness



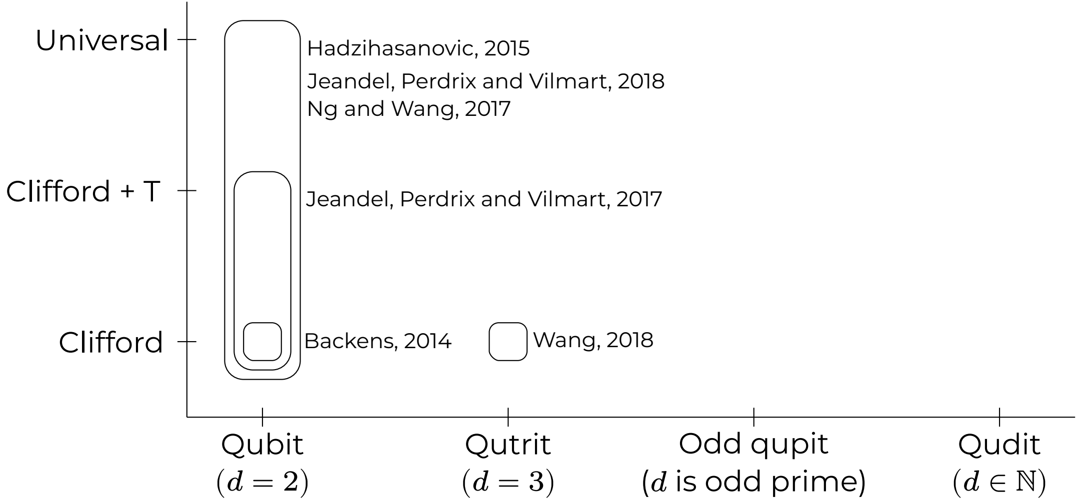
History of Completeness



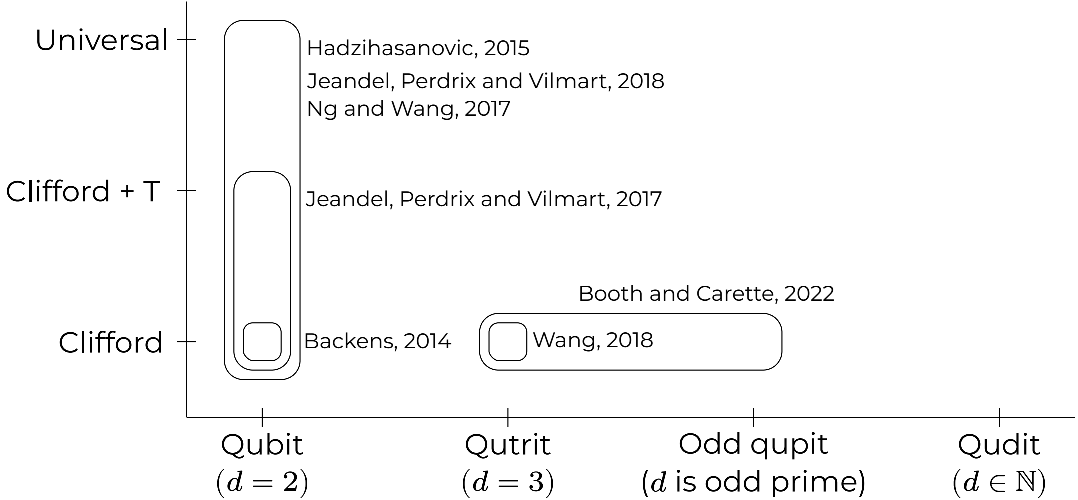
History of Completeness



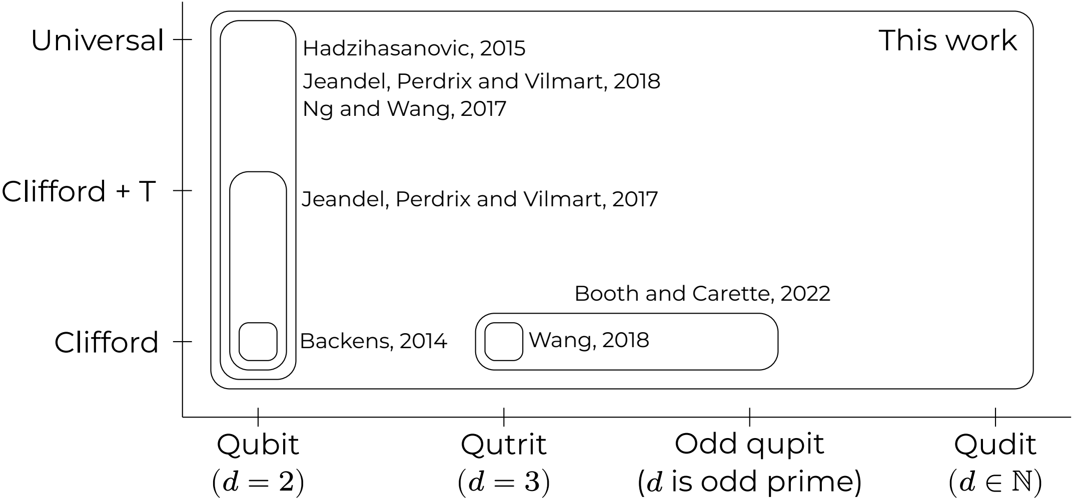
History of Completeness



History of Completeness



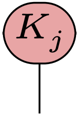
History of Completeness



The qudit ZXW-calculus

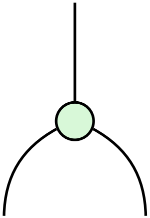
Standard basis in qudit ZXW

For $0 \leq j < d$,



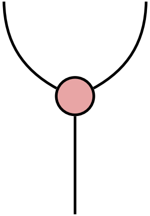
$$\xmapsto{[\![\cdot]\!]} \quad |d - j\rangle$$

Z spider



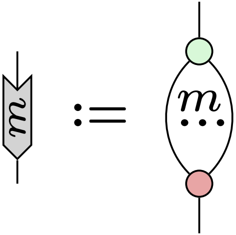
$$\xrightarrow{[\![\cdot]\!]} \quad |k\rangle \mapsto |k,\ k\rangle$$

X spider

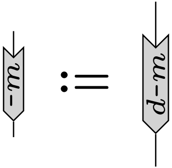


$$\xrightarrow{[\![\cdot]\!]} \quad |i,\ j\rangle \mapsto |i + j \mod d\rangle$$

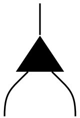
Notation: The multiplier



$$\xrightarrow{[\![\cdot]\!]} \quad |k\rangle \mapsto |m \cdot k \mod d\rangle \,,$$

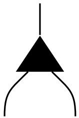


Generator: W node

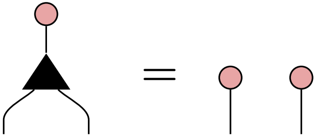


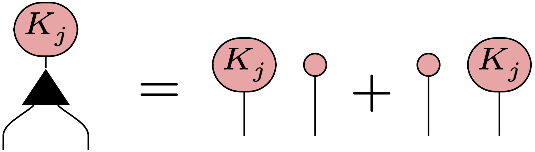
$$\overset{[\![\cdot]\!]}{\longmapsto} \quad |00\rangle\langle 0| + \sum_{i=1}^{d-1}(|i0\rangle + |0i\rangle)\langle i|$$

Generator: W node



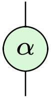
$$\xmapsto{[\![\cdot]\!]} \quad |00\rangle\langle 0| + \sum_{i=1}^{d-1} \left(|i0\rangle + |0i\rangle \right) \langle i|$$





Understanding the Z box

Z spider:



$$\xrightarrow{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix}, \qquad \text{where } \alpha \in \mathbb{R}.$$

Z box:

$$\xrightarrow{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}, \qquad \text{where } a \in \mathbb{C}.$$

Understanding the qudit Z box

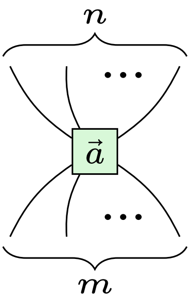
Qubit Z box: for $a \in \mathbb{C}$,

$$\xmapsto{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$$

Qudit Z box: for $\vec{a} = (a_1, a_2, \cdots, a_{d-1}) \in \mathbb{C}^{d-1}$,

$$\xmapsto{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{d-1} \end{bmatrix}$$

Generator: Z box

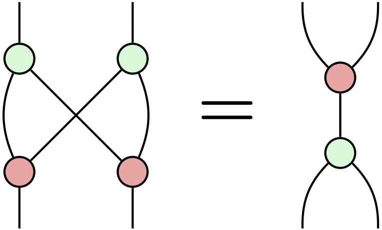


$$\xrightarrow{[\![\cdot]\!]} \sum_{j=0}^{d-1} a_j \left|j\right\rangle^{\otimes m} \left\langle j\right|^{\otimes n},$$

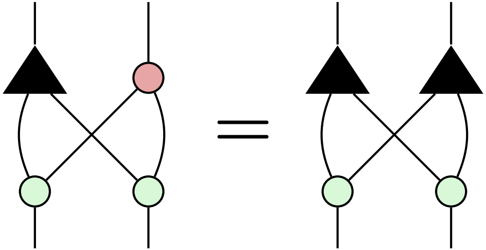
where $\vec{a} = (a_1, \cdots, a_{d-1}) \in \mathbb{C}^{d-1}$

and $a_0 := 1$

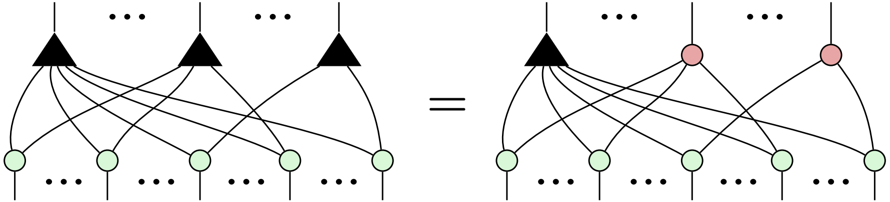
Rule: Bialgebra



Rule: Trialgebra

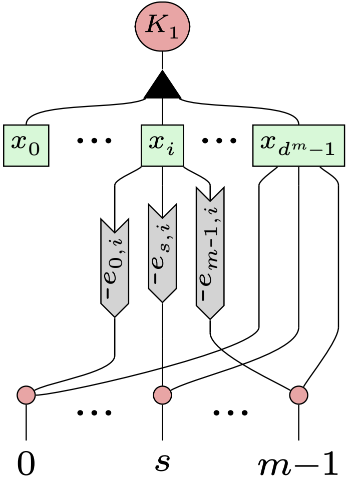


Generalised Trialgebra



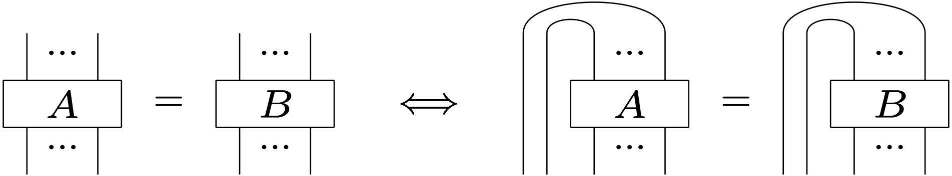
A Normal Form

$$\begin{pmatrix} x_0 \\ \vdots \\ x_i \\ \vdots \\ x_{d^m-1} \end{pmatrix} \longrightarrow$$



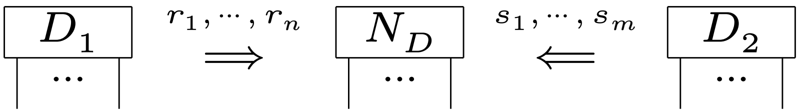
Completeness proof

Map-state duality



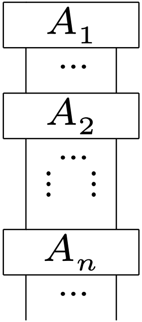
Completeness using a normal form

If $[\![D_1]\!] = [\![D_2]\!]$, then:

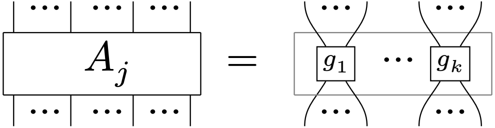


Note: Structure of states

Each state diagram has the following structure:

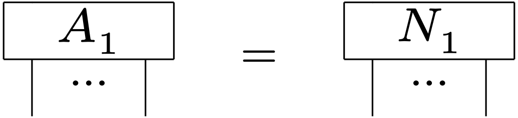


with

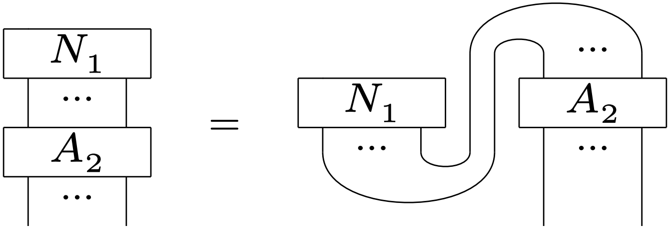


where $g_1, \cdots, g_k$ are generators.

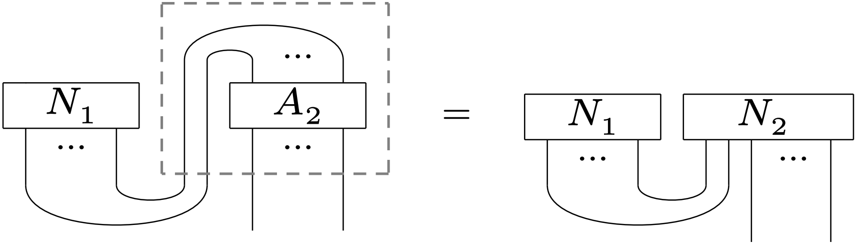
State $\Longrightarrow$ normal form I.



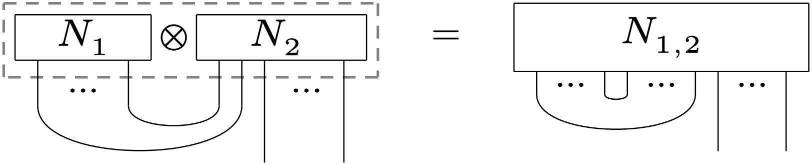
State $\implies$ normal form II.



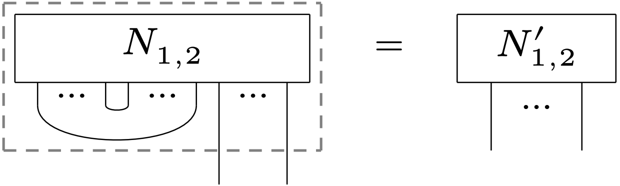
State $\implies$ normal form III.



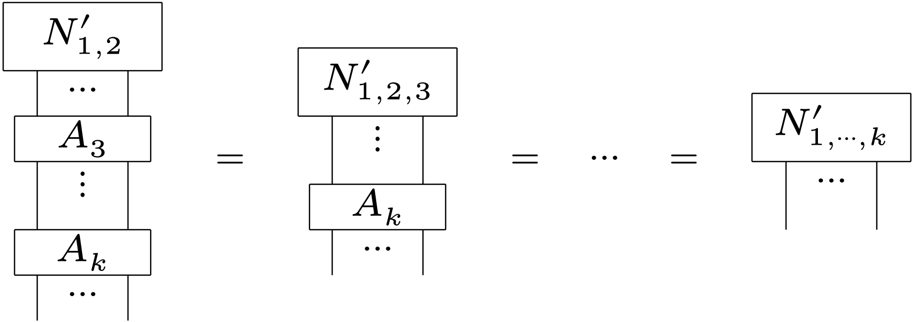
State $\Longrightarrow$ normal form IV.



State $\Longrightarrow$ normal form V.



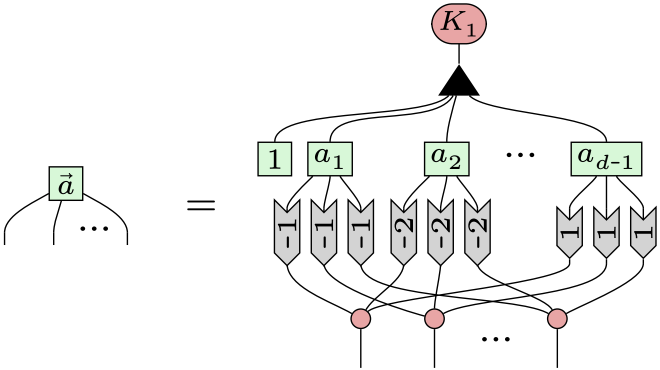
State $\implies$ normal form VI.



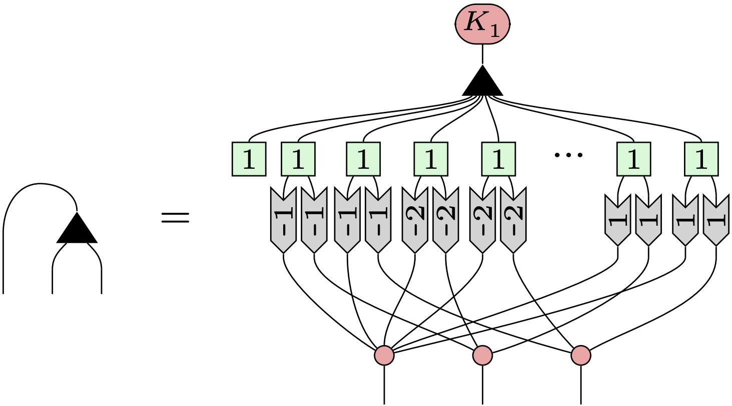
Summary: state $\Longrightarrow$ normal form

- Generators
- Tensor product of two normal forms
- Partial-traced normal form

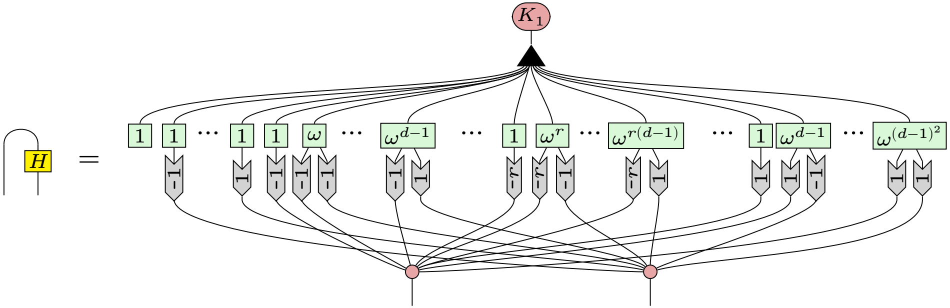
Lemma: Z box



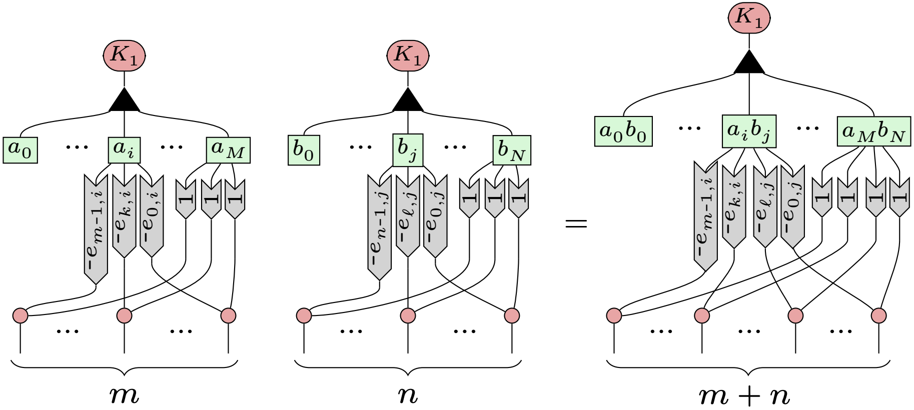
Lemma: W node



Lemma: Hadamard box

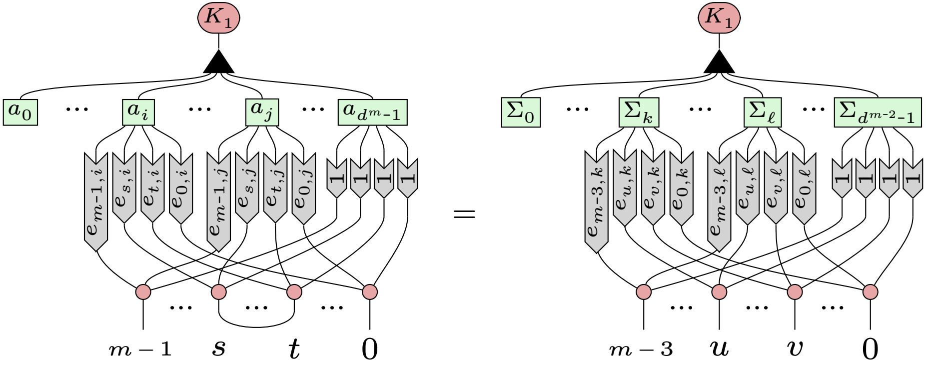


Lemma: Tensor product



where $M = d^m - 1,\ N = d^n - 1.$

Lemma: Partial trace



where Σ_k is the sum of those elements where the indices match.

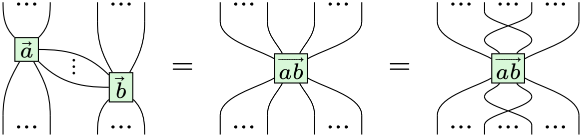
Axioms of ZXW-calculus

1. Rules of ZX

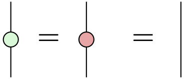
2. Rules of ZW

3. Rules of ZXW

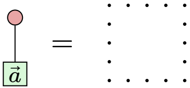
The ZX-part of the rules I



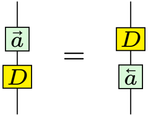
(S1)



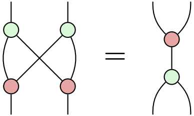
(S2)



(Ept)



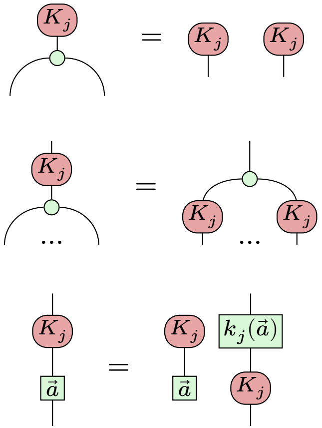
(D1)



(B2)

where $\overleftarrow{a} = (a_{d-1}, \dots, a_1)$, $\overrightarrow{ab} = (a_1 b_1, \dots, a_{d-1} b_{d-1})$.

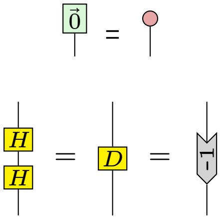
The ZX-part of the rules II



(K0)

(K1)

(K2)

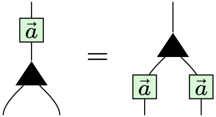


(Zer)

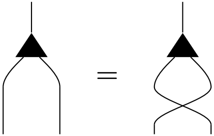
(P1)

where $k_j(\vec{a}) = \left(\frac{a_{1-j}}{a_{d-j}}, \dots, \frac{a_{d-1-j}}{a_{d-j}}\right)$

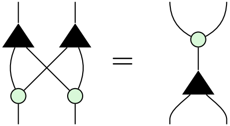
The ZW-part of the rules



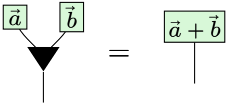
(Pcy)



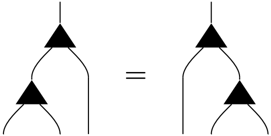
(Sym)



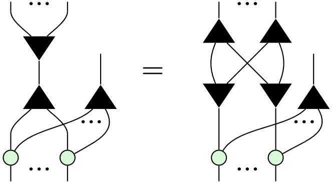
(BZW)



(AD)

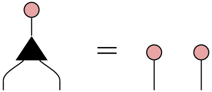


(Aso)

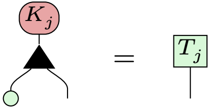


(WW)

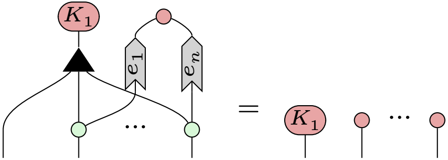
The ZXW-part of the rules I



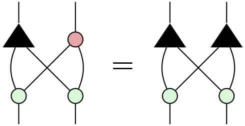
(Bs0)



(Bsj)



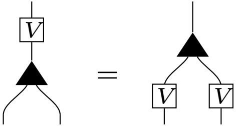
(KZ)



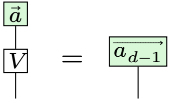
(TA)

where $T_j = \overbrace{(\underbrace{0, \dots, 1}_{d-j}, \dots, 0)}^{d-1}$, $\quad e_1, \dots, e_n \in \{1, \dots, d-1\}$.

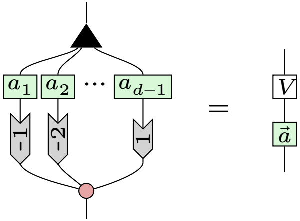
The ZXW-part of the rules II



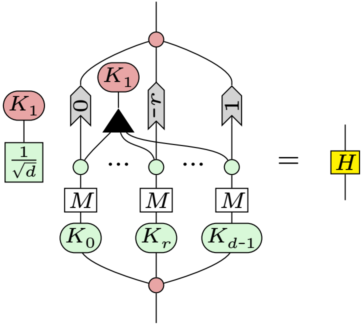
(VW)



(ZV)



(VA)



(HD)

where $\overrightarrow{a_{d-1}} = (a_{d-1}, a_{d-1}, \dots, a_{d-1})$.

Outlook

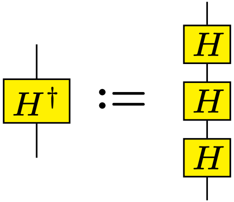
- *Light-matter interaction in the ZXW calculus*
  Talk today at 14:30
- Optimisation of qudit circuits
- Completeness of qudit ZX-calculus
- Completeness of qufinite ZXW-calculus
- Hamiltonian simplification with ZXW

Appendix

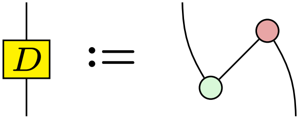
Overview

1. Introduction
   - ZXW-calculus
   - Qudits
   - Completeness
2. The qudit ZXW-calculus
   - Genarators
3. Completeness proof
   - Proof idea
   - Lemmas
4. Axioms of ZXW

Notation: The Hadamard inverse

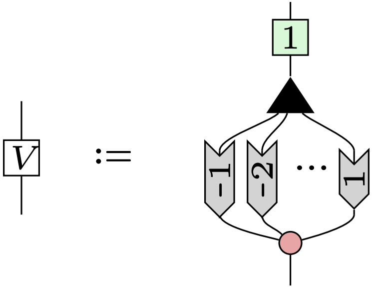


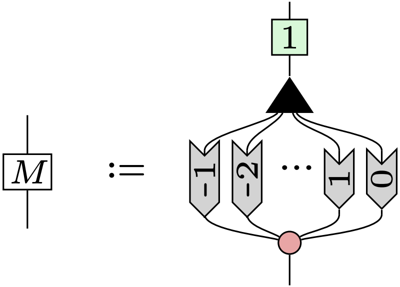
Notation: The dualiser



$$\overset{[\![\cdot]\!]}{\longmapsto} \sum_{i=0}^{d-1} |i\rangle \langle -i| \,.$$

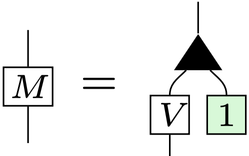
Notation: The V and M boxes





with

$$\xmapsto{[\![\cdot]\!]} \quad |0\rangle\langle 0| + \sum_{i=1}^{d-1} |i\rangle\langle -1|$$



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