Contextuality with vanishing coherence and maximal robustness to dephasing

Vinicius P. Rossi, David Schmid, John H. Selby, Ana Belén Sainz

International Centre for Theory of Quantum Technologies, University of Gdańsk

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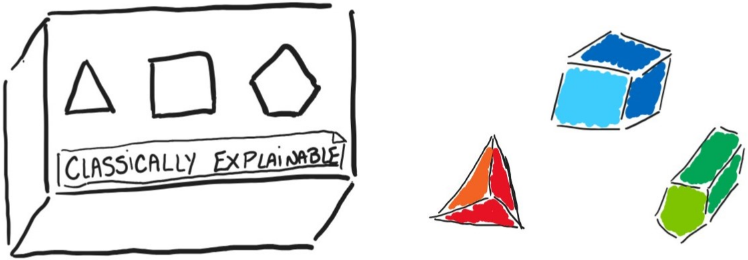
Outline

Motivation



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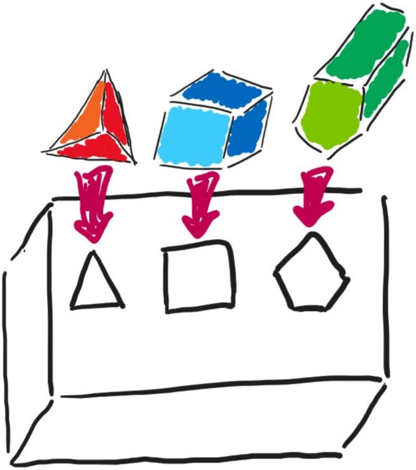

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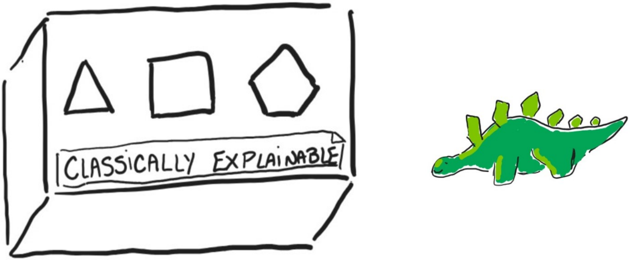

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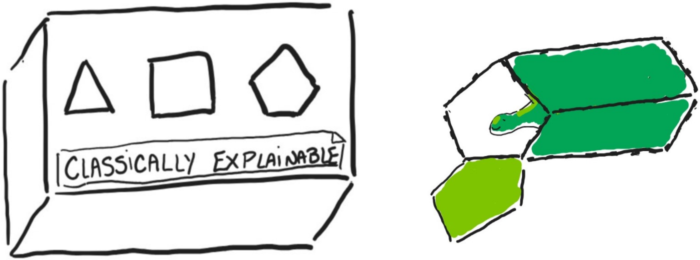

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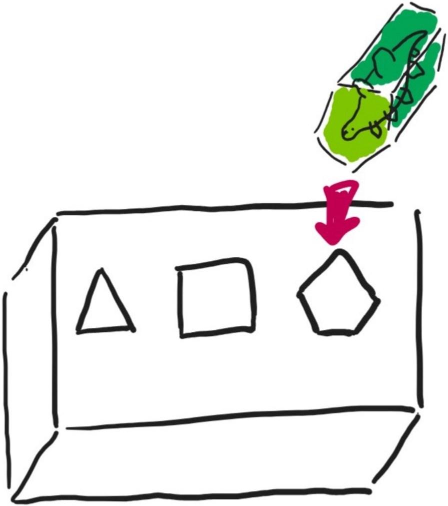


Motivation

Definitions

- **Operational prepare and measure scenario:** a tuple $(\mathcal{P}, \mathcal{M}, p)$, where
  - $\mathcal{P}$ = possible preparations;
  - $\mathcal{M}$ = possible measurement outcomes for each measurement;
  - p = a rule on how to compute probabilities for each measurement outcome conditioned to each preparation, $\{p(k|M,P)\}_{P\in\mathcal{P},[k|M]\in\mathcal{M}}$;
- **Equivalence classes:** a subset $e(P) \subset \mathcal{P}$ such that it can be defined as

$$e(P) := \{P' \in \mathcal{P} | p(k|M,P') = p(k|M,P), \quad \forall [k|M] \in \mathcal{M}\}, \tag{1}$$

with a similar definition for measurement equivalence classes.[1]

[1] I'm assuming tomographic completeness.

Definitions

- **Ontological model associated to $(\mathcal{P}, \mathcal{M}, p)$:** a tuple $(\Lambda, \{\mu(\lambda|P)\}_{\lambda\in\Lambda, P\in\mathcal{P}}, \{\xi(k|M,\lambda)\}_{\lambda\in\Lambda,[k|M]\in\mathcal{M}})$, where
  - Λ is a measurable space;
  - $\mu(\lambda|P)$ and $\xi(k|M,\lambda)$ are conditional probability distributions;
  - $p(k|M,P) = \int_{\lambda\in\Lambda} \xi(k|M,\lambda)\mu(\lambda|P)d\lambda$, for all $[k|M] \in \mathcal{M}$, $P \in \mathcal{P}$.
- **Generalised non-contextuality:** The assumption that for all $\lambda \in \Lambda$
  - $\mu(\lambda|P) = \mu(\lambda|e(P)), \forall P \in \mathcal{P}$;
  - $\xi(k|M,\lambda) = \xi(e(k|M),\lambda), \forall [k|M] \in \mathcal{M}$.

Definitions

- **GPT associated to an operational scenario:** A tuple $(V, \Omega, \mathcal{E})$ where
  - V is a finite real inner product space;
  - $\Omega \in V$ is a convex set such that each $e(P) \in \mathcal{P}$ is mapped to a vector $s \in \Omega$;
  - $\mathcal{E} \subseteq \Omega^*$ is a convex set such that each $e(k|M) \in \mathcal{M}$ is mapped to a vector $e \in \mathcal{E}$;
  - $p(k|M, P) = \langle e, s \rangle$.
- **Strictly classical scenarios:** A scenario is strictly classical when its associated GPT is *simplicial*, i.e., $\Omega = \Delta_d$ a d-dimensional simplex and $\mathcal{E} = \Delta_d^*$, its dual hypercube.

Simplex embedding

Simplex embeddable scenarios

A GPT $(V, \Omega, \mathcal{E})$ simplex embeddable if there is a simplex $\Delta_d \subset \mathbb{R}^n$, and linear maps $\iota : \Omega \to \Delta_d$, $\kappa : \mathcal{E} \to \Delta_d^*$ such that

- $\iota(\Omega) \subseteq \Delta_d$;
- $\kappa(\mathcal{E}) \subseteq \Delta_d^*$;
- Inner products are preserved by these maps.

If this GPT is associated to an operational scenario, then the scenario admits of a noncontextual ontological model.[2]

[2] PRX Quantum 2, 010331

Fact 1

Incoherent states/effects cannot proof contextuality.

Proof. $\mathcal{P} = \{\rho_P = \sum_{i=1}^{d} q_i^P |i\rangle\langle i|\}_{P\in\mathcal{P}}$, $\mathcal{M} = \{E_{k|M}\}_{[k|M]\in\mathcal{M}}$. Define

$$\mu(i|P) := \langle i|\rho_P|i\rangle \,, \quad \xi(k|M,i) := \mathrm{Tr}\{E_{k|M} |i\rangle\langle i|\}. \tag{2}$$

$$\begin{aligned} \mathrm{Tr}\{E_{k|M}\rho_P\} &= \sum_{i\in I} \langle i|E_{k|M}\rho_P|i\rangle & (3)\\ &= \sum_{i\in I} \langle i|E_{k|M}|i\rangle \langle i|\rho_P|i\rangle & (4)\\ &= \sum_{i\in I} \xi_{k|M}(i)\mu_P(i), & (5) \end{aligned}$$

which is a noncontextual model by the definitions of $\mu(i|P)$ and $\xi(k|M,i)$.

Fact II

Any quantum scenario admits of a simplex embedding under finite partial depolarising noise.[3]

$$\mathcal{D}_{\text{depol}}[\rho] = (1-r)\rho + \frac{r}{2}\mathbb{1}. \tag{6}$$

[3]Phys. Rev. X 8, 011015; Phys. Rev. Lett. 115, 110403; arXiv:2003.05984.

Question

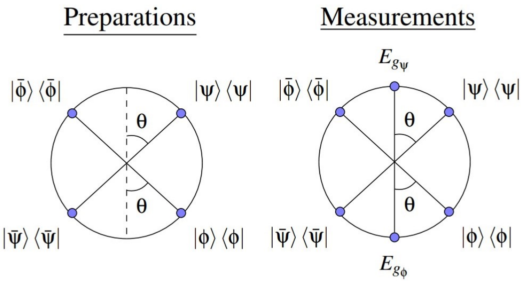
Since coherence is necessary for contextuality proofs and a finite amount of partial depolarising noise is enough to destroy proofs of contextuality, what happens when we take dephasing into consideration?

$$\mathcal{D}_{\text{deph}}[\rho] = (1-r)\rho + r\sum_{i=0}^{1} |i\rangle\langle i|\rho|i\rangle\langle i| \,. \tag{7}$$

Selby, Wolfe, Schmid, Sainz, arXiv:2204.11905 (2022)

1. Asks for a set of states Ω, a set of effects $\mathcal{E}$, a unit effect $\mathbb{1}$ and a maximally mixed state μ;
2. Finds the minimal real vector space V in which both Ω and $\mathcal{E}$ can live and computes their cone facets H_Ω, $H_\mathcal{E}$ and the inclusion matrices I_Ω and $I_\mathcal{E}$ onto this vector space;
3. Solves the linear program

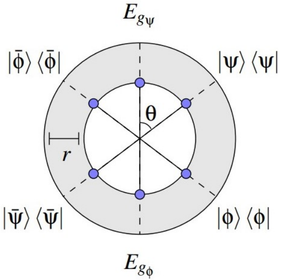
$$\begin{aligned} \min \quad & r \\ \text{s.t.} \quad & r I_\mathcal{E}^T \cdot \mathcal{D} \cdot I_\Omega + (1-r) I_\mathcal{E}^T \cdot I_\Omega = H_\mathcal{E}^T \cdot \sigma \cdot H_\Omega, \\ & \sigma \geq_e 0\,. \end{aligned} \tag{8}$$



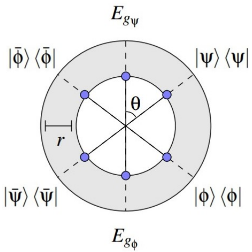
Here θ is proportional to most quantifiers for coherence (e.g. trace distance $C(\theta) = \sin\theta$)

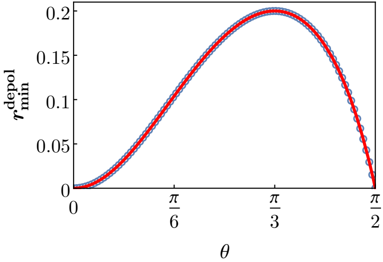
Schmid & Spekkens, Phys. Rev. X 8, 011015

Depolarising noise



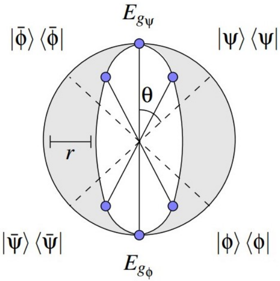
Result 1



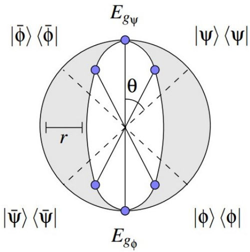


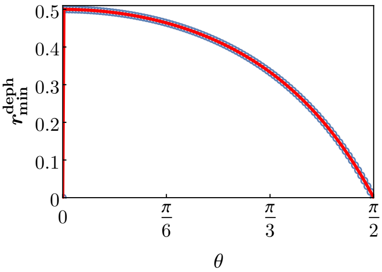
$r > 0$ for all $0 < \theta < \frac{\pi}{2}$ for the embedding to exist $\implies$ **proof of contextuality for any amount of coherence.**

Dephasing noise



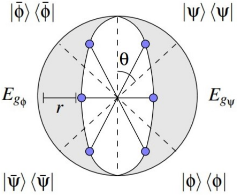
Dephasing noise

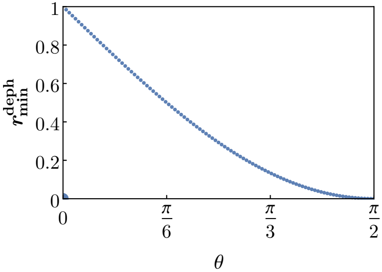




Smaller the coherence, higher the robustness. Can we make it maximally robust?

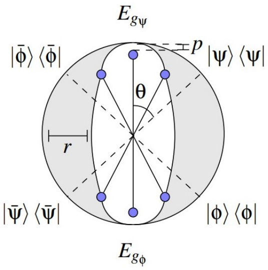
Result 2

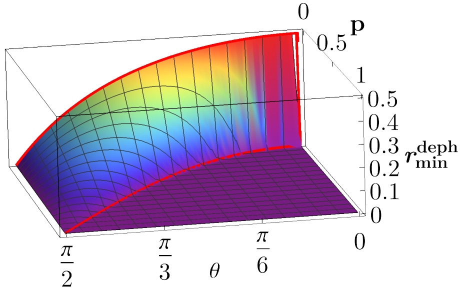




Maximal robustness to dephasing for vanishing coherence $\implies$ **There are scenarios that will not admit of a simplex embedding by any amount of partial dephasing noise.**

Result 3





Clear cut between θ and p on when a proof of contextuality exists or not. If the coherence quantifier is the trace distance $C(\theta) = \sin\theta$, then

$$C(\theta) \geq \sqrt{p(2-p)}. \quad (9)$$

Final message

- **Summary:** Any scenario will admit of a simplex embedding after partial depolarisation, but we show that the same is not true for dephasing. Therefore, there are proofs of contextuality maximally robust to dephasing.
- **Why does it matter?** dephasing is one of the most common noises in quantum computing, and finding proofs of nonclassicality maximally robust to it can yield applications;
- **Future challenges:** how to talk about dephasing beyond the rebit GPT? How to generalise the code for other types of noise? Is there such a maximal robustness for scenarios other than this family?

Acknowledgements

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