Inequalities witnessing coherence, nonlocality, and contextuality

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Introduction

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- **nonclassical** features of quantum theory
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Can we understand the interplay between them?

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- generalises basis-independent coherence witnesses
- recovers all noncontextuality inequalities from the CSW approach
- also related to preparation contextuality in a specific setup

Event graph approach

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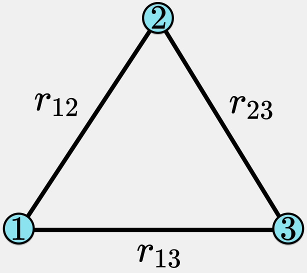
Take a graph G (event graph).



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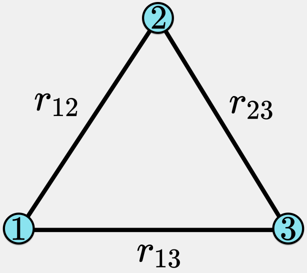

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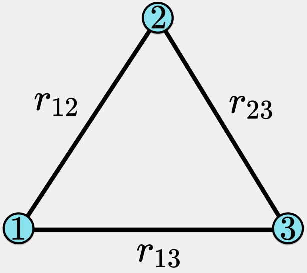
- Edge weight $r_{ij} = \text{Prob}(A_i = A_j)$
- Note: in dichotomic case $\Lambda = \{-1, +1\}$, $\langle A_i A_j \rangle = 2r_{ij} - 1$.



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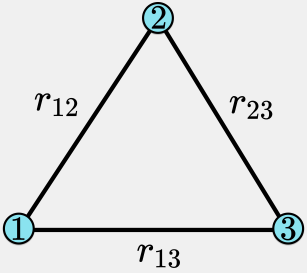
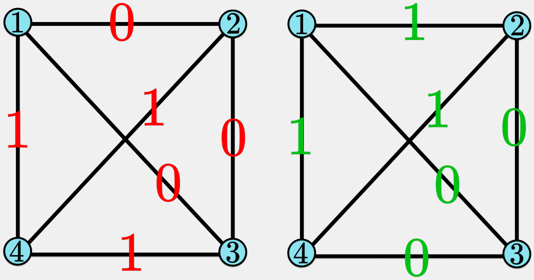


An edge weighting $r : E(G) \longrightarrow [0,1]$ is **classical**
if it arises in this fashion from jointly distributed $\{A_i\}_{i \in V(H)}$.

$\rightsquigarrow$ **Classical polytope** $C_G \subseteq [0,1]^{E(H)}$.

Vertices of the classical polytope

- Vertices of C_G are deterministic edge-labellings $\alpha : E(G) \longrightarrow \{0, 1\}$



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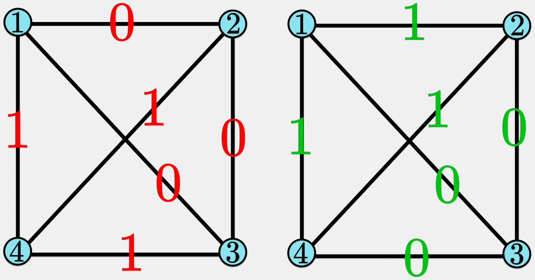
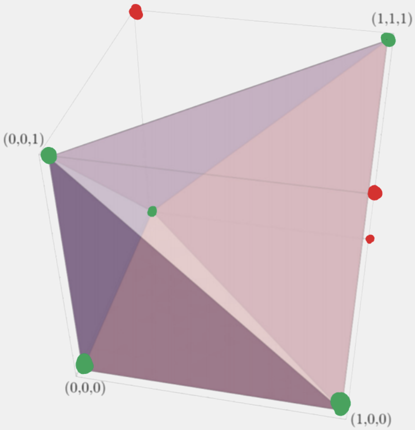
- arising from underlying vertex labelling $V(H) \longrightarrow \Lambda$
  with 1 meaning $=$, 0 meaning $\neq$

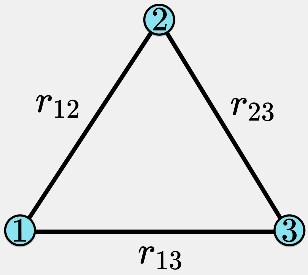


Allowed labellings are those that do not violate the **transitivity of equality**

The classical polytope

Forbidden (1,1,0) (1,0,1) (0,1,1)
Allowed (0,0,0) (1,1,1) (0,0,1) (0,1,0) (1,0,0)





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$$A_1 = A_2 \qquad A_2 = A_3 \qquad A_1 \neq A_3$$

yield inequality

$$\Pr(A_1 = A_2) + \Pr(A_2 = A_3) + \Pr(A_1 \neq A_3) \leq 2$$

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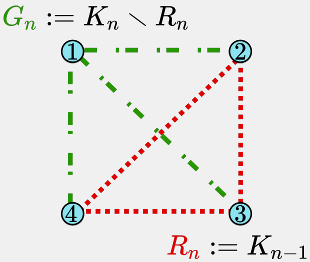
$$\begin{aligned} & \Pr(A_1 = A_2) + \Pr(A_2 = A_3) + \Pr(A_1 \neq A_3) \leq 2 \\ \Leftrightarrow \\ & \Pr(A_1 = A_2) + \Pr(A_2 = A_3) + (1 - \Pr(A_1 = A_3)) \leq 2 \\ \Leftrightarrow \\ & \Pr(A_1 = A_2) + \Pr(A_2 = A_3) - \Pr(A_1 = A_3) \leq 1 \\ \Leftrightarrow \\ & r_{12} + r_{23} - r_{13} \leq 1 \end{aligned}$$

Classical polytope inequalities

- Cycle inequalities (Brod–Galvão arXiv:1902.11039 [quant-ph])

$\sum_{i=1}^{n-1} r_{i,i+1} - r_{1n} \leq n-2$



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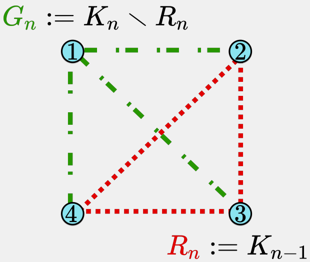
  $(r_{12} + r_{13} + r_{14}) - (r_{23} + r_{34} + r_{24}) \leq 1$



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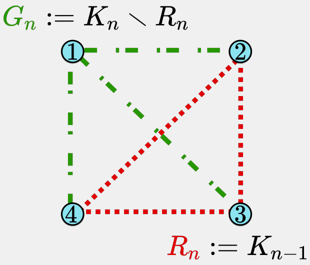

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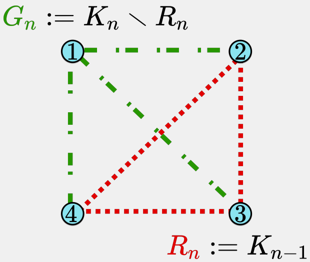
  $G_n := \{\{1,i\} \mid i = 2,\dots,n\} \quad R_n := E(K_n) \setminus G_n$

  $\sum_{e\in G_n} r_e - \sum_{e\in R_n} r_e = k - \sum_{e\in R_n} r_e \leq k - \binom{k}{2} = 1 - \binom{k-1}{2} \leq 1$



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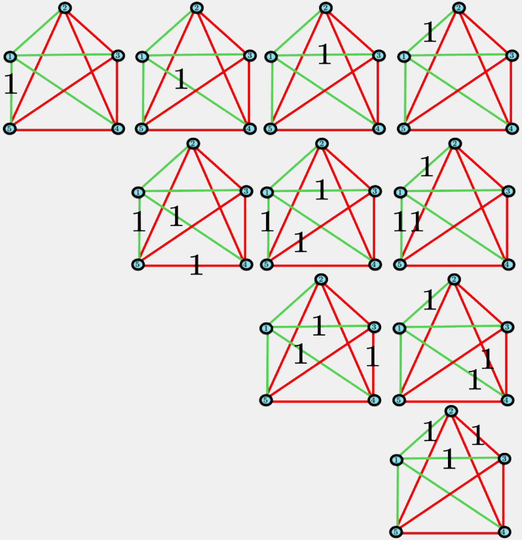

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- We are interested in a basis independent notion
- Relational property of a **set** of states
- A set os states is coherence-free if these can be simultaneously diagonalised

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Set of states $\{|\phi_i\rangle\}_{i \in V(H)}$ and consider overlaps $r_{ij} = |\langle\phi_i|\phi_j\rangle|^2 = \mathrm{Tr}(\rho_i \rho_j)$.

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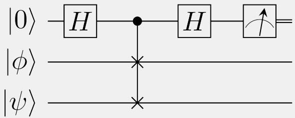

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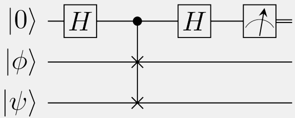

If coherence-free $\rho = \begin{pmatrix} \rho_{11} & 0 & 0 \\ 0 & \ddots & \\ 0 & 0 & \rho_{dd} \end{pmatrix}$ $\sigma = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \ddots & \\ 0 & 0 & \sigma_{dd} \end{pmatrix}$

then $\mathrm{Tr}(\rho\sigma) = \sum_i \rho_{ii}\sigma_{ii} = \sum_{i=j} \rho_{ii}\sigma_{jj}$

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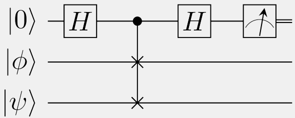

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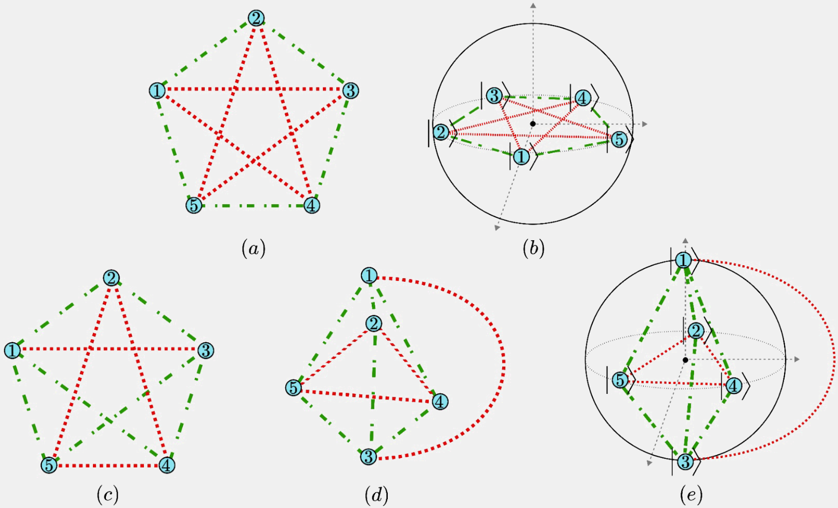
Any $r \in C_G$ admits realisation by coherence-free set of states

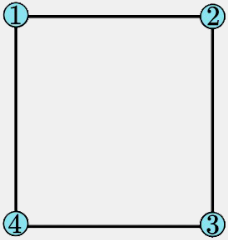
Quantum violations



Nonlocality and contextuality

CHSH inequality

- Cycle inequality $r_{12} + r_{23} + r_{34} - r_{14} \ \leq \ 2$



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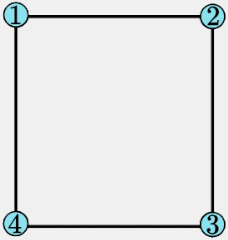

- Cycle inequality $r_{12} + r_{23} + r_{34} - r_{14} \leq 2$
- Interpret vertices as Alice's or Bob's local measurements:
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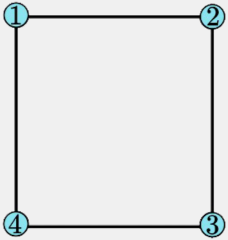

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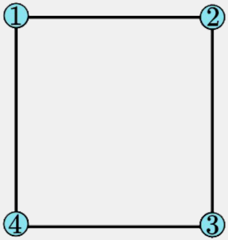

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- Measuring on singlet state: $r_{AB} = p_{\neq}^{AB} = 1 - p_{=}^{AB}$
- So the facet inequality is rewritten as

$$p_{\neq}^{A_1B_1} + p_{\neq}^{A_2B_1} + p_{\neq}^{A_2B_2} - p_{\neq}^{A_1B_2} \;\leq\; 2.$$

CHSH inequality

CSW approach: exclusivity graphs

Take a graph H, interpreted as **exclusivity** graph:

- vertices: measurement events
- edges: exclusive events (distinguishable by a measurement)

In quantum mechanics:

- vertices: projectors (PVM elements)
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Consider assignments of probabilities to events $V(H) \longrightarrow [0, 1]$.

CSW approach: noncontextual polytope

Deterministic assignments $V(H) \longrightarrow \{0,1\}$ – equivalently, subsets of $V(H)$.

Which are valid truth-values assignments?

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Deterministic assignments $V(H) \longrightarrow \{0, 1\}$ – equivalently, subsets of $V(H)$.

Which are valid truth-values assignments?

- $S \subseteq V(H)$ is **stable** if no two vertices are adjacent
- Take $\chi_S : V(H) \longrightarrow \{0, 1\}$

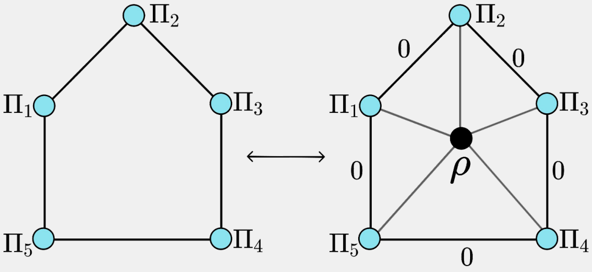
- Stability indicates that exclusive measurement events cannot be simultaneously true

Noncontextual polytope $\mathrm{STAB}(H) \subseteq [0, 1]^{V(H)}$:

$$\mathrm{STAB}(H) := \mathrm{ConvHull}\left\{\chi_S \in [0, 1]^{V(H)} \mid S \subseteq V(H) \text{ stable}\right\}.$$

Recovering the noncontextual polytope

Start with a graph H, thought of as an exclusivity graph (in CSW sense)



Recovering the noncontextual polytope

Start with a graph H, thought of as an exclusivity graph (in CSW sense)

Define a new graph H_* by adjoining a new vertex connected to every existing vertices:

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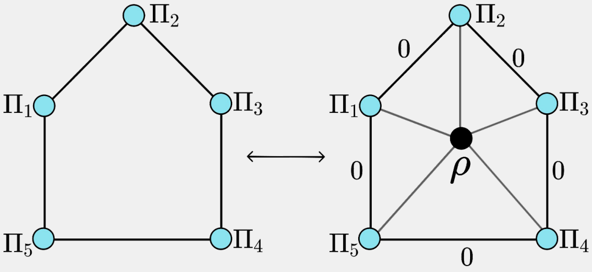

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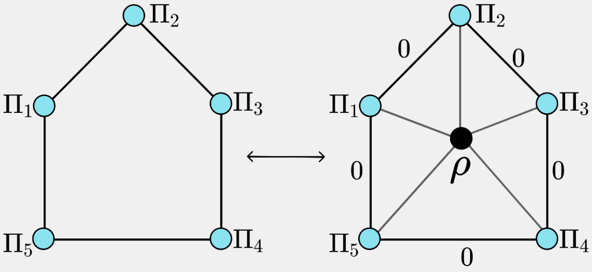
Impose overlap 0 on the edges of H.



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$$C^0_{H^*} := \{r \in C_H \mid \forall e \in E(H).\ r_e = 0\}$$

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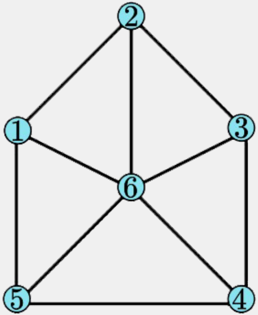
Noncontextuality inequalites obtained from C_{H^*} ineqs by setting $E(H)$ coefficients to zero.

Recovering noncontextality ineqalities

6-vertex wheel graph W_6

C_{W_6} has a facet-defining inequality:

$$-r_{12} - r_{23} - r_{34} - r_{45} - r_{15} + r_{16} + r_{26} + r_{36} + r_{46} + r_{56} \;\leq\; 2$$



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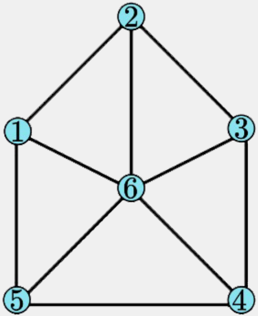

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- Neighboring vertices in outer 5-cycle: orthogonal projectors
- $r_v6 =$ probability of successful projection of the central vertex state onto the projector associated with vertex v.

Recovering noncontextality ineqalities

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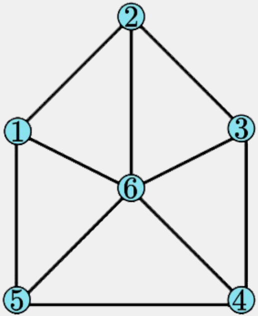

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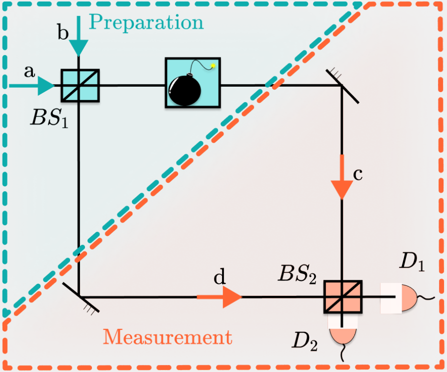
- $r_v 6 =$ probability of successful projection of the central vertex state onto the projector associated with vertex v.

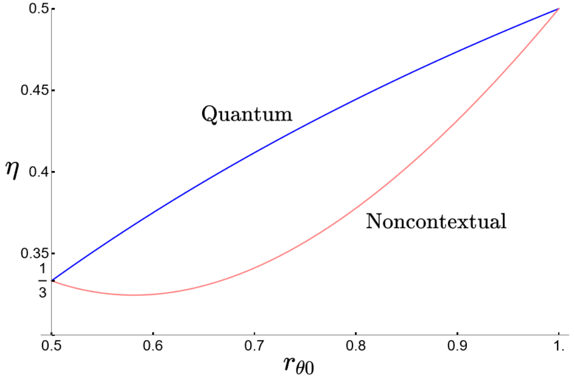
Imposing exclusivity constraints $r_{ij} = 0$ in the outer cycle yields the inequality

$$r_{16} + r_{26} + r_{36} + r_{46} + r_{56} \leq 2,$$

KCBS inequality

Application: quantum interrogation in MZ interferometer





Questions...

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