Admissible Causal Structures and Correlations

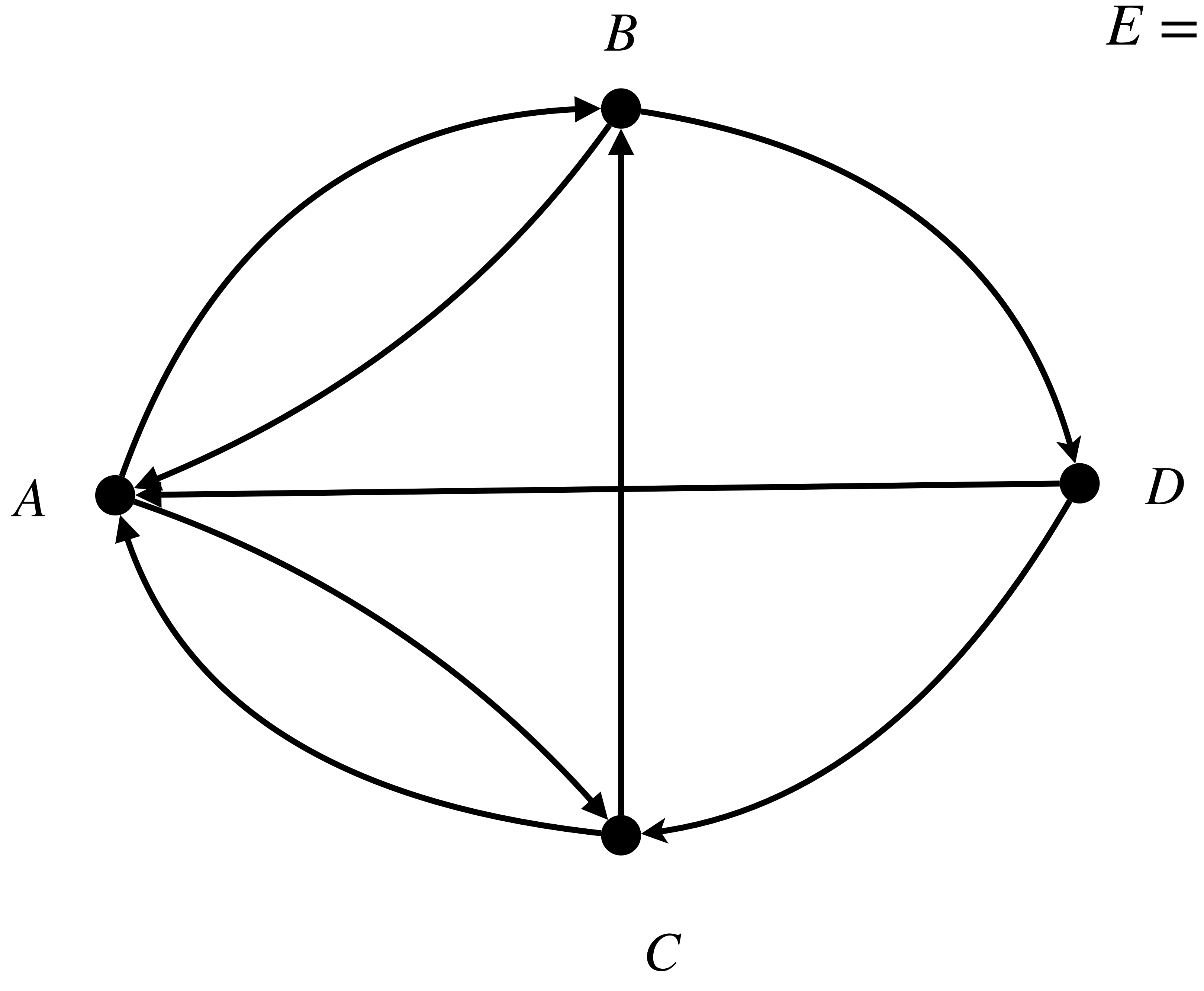
E.T., joint work with Ämin Baumeler

QPL 2023

YIRG Vienna

IQI Vienna

arXiv:2210.12796



$$G = (V, E)$$

$$V = \{A, B, C, D\}$$

$$E = \{(A, B), (B, A), (A, C), (C, A), (C, B), (B, D), (D, C)\}$$

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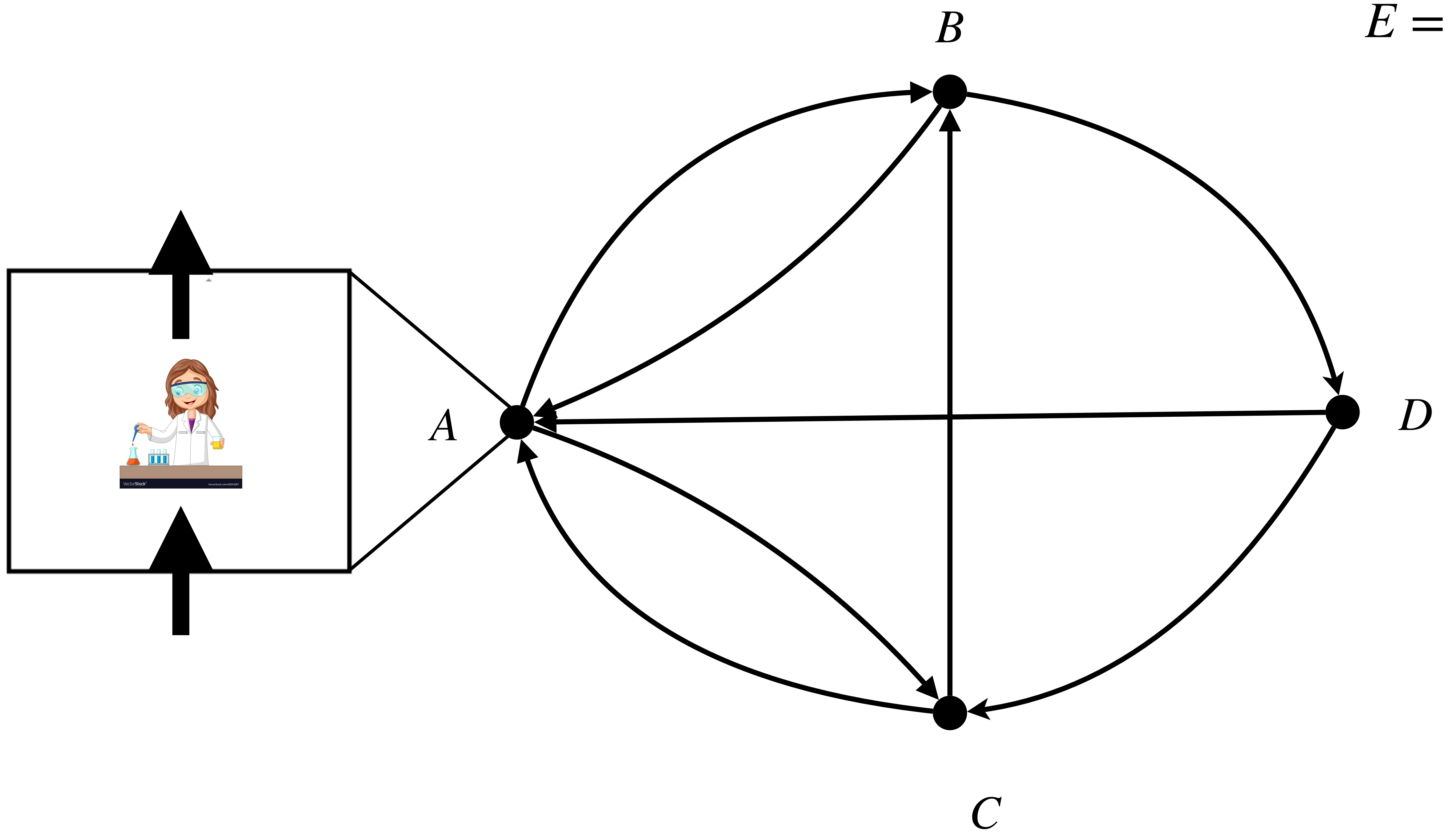

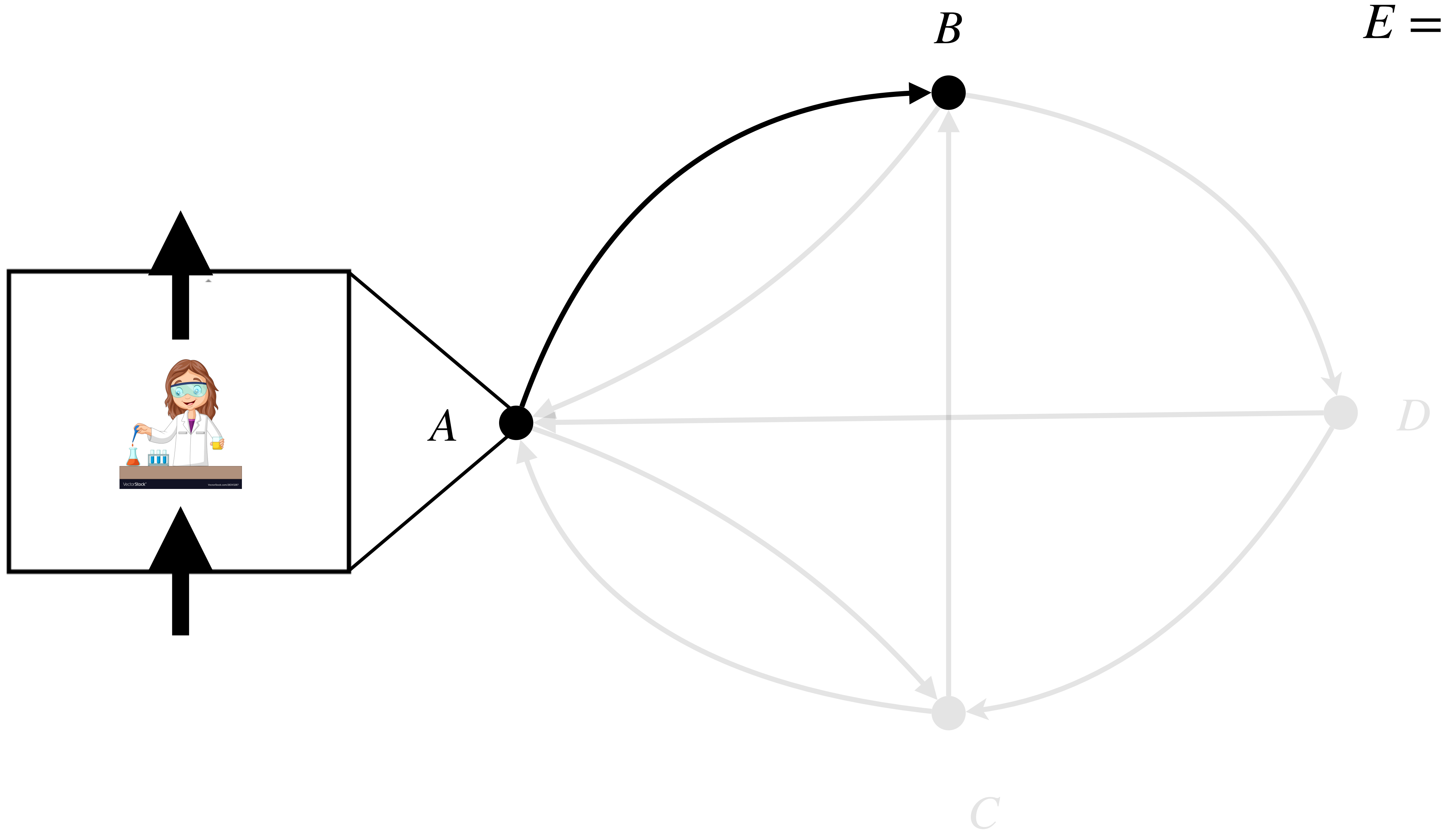

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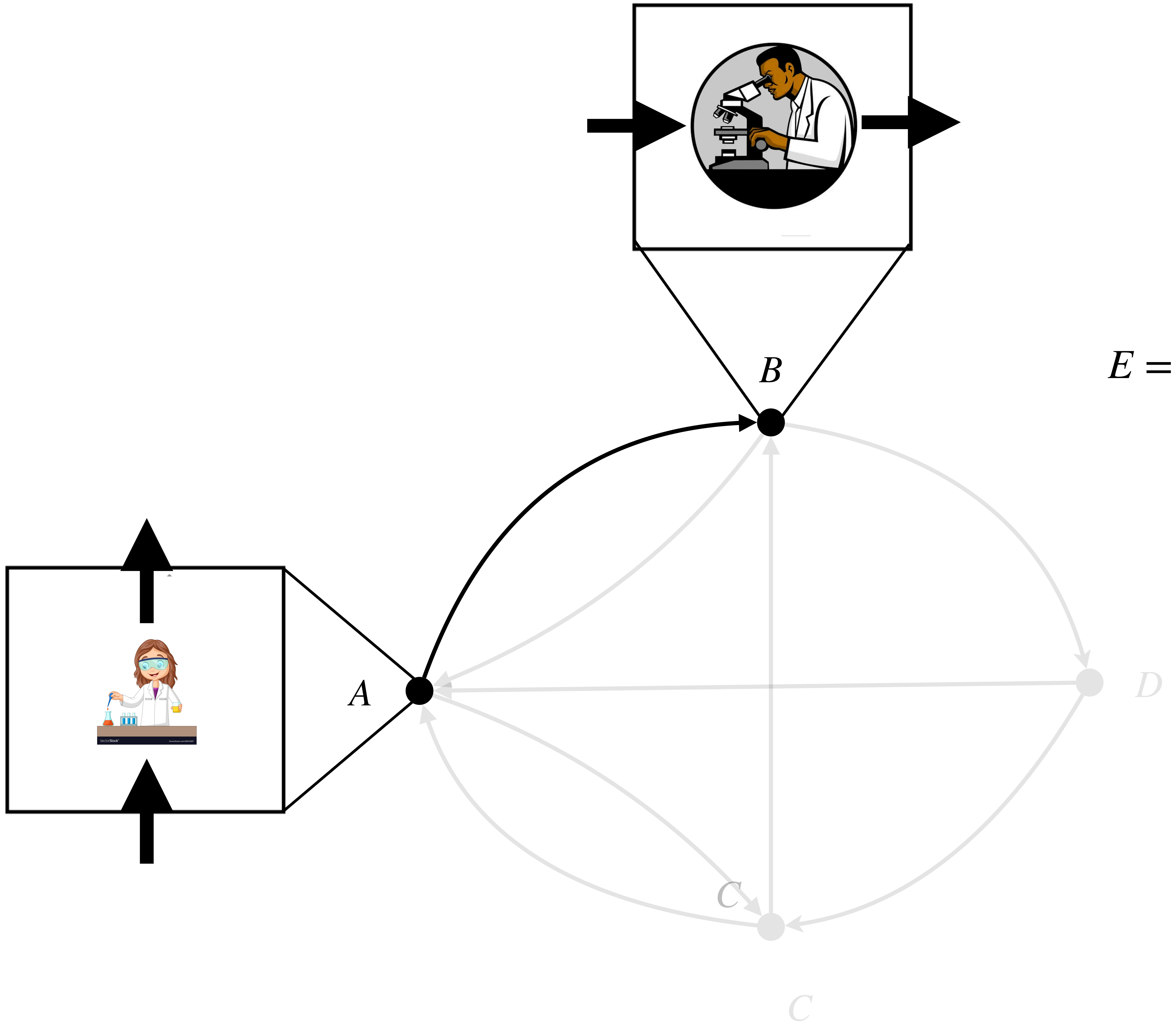

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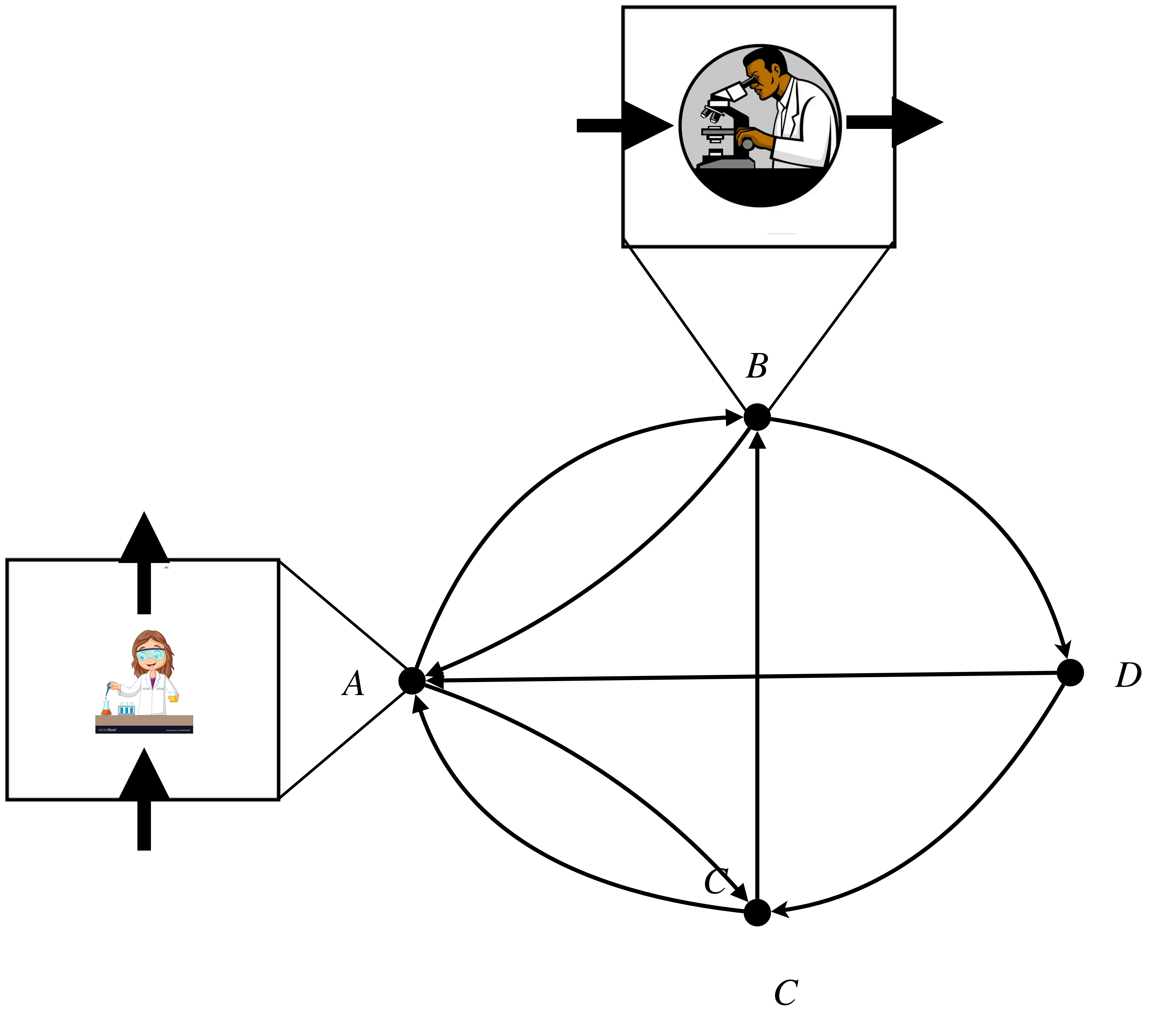

Qualitative limitations on

·Causal structures

·Correlations

under the assumption of local quantum theory

Motivation

Classical vs Quantum Causal Structures

Does quantum theory allow for more general causal structures than classical?

Information Processing

New forms of communication protocols, e.g., local operations and classical non-causal communication, R. Kunjwal and Ä. Baumeler, arXiv:2202.00440.

Computational power of non-causal processes

What is the computational power of this framework, Ä. Baumeler, S. Wolf arXiv:1611.05641.

Relation to physical theories

How can we implement such scenarios in physical settings, e.g., general relativity or quantum gravity.
Information is physical
Ä. Baumeler, F. Costa, T. C Ralph, S. Wolf and M. Zych
(2019) *Class. Quantum Grav.* **36** 224002
NS Móller, B Sahdo, N Yokomizo, arXiv:2306.10984

Outline

- *Preliminaries*
  - *Process matrices*
  - *Causal models*
- *Admissible Causal Structures*
- *Correlations*
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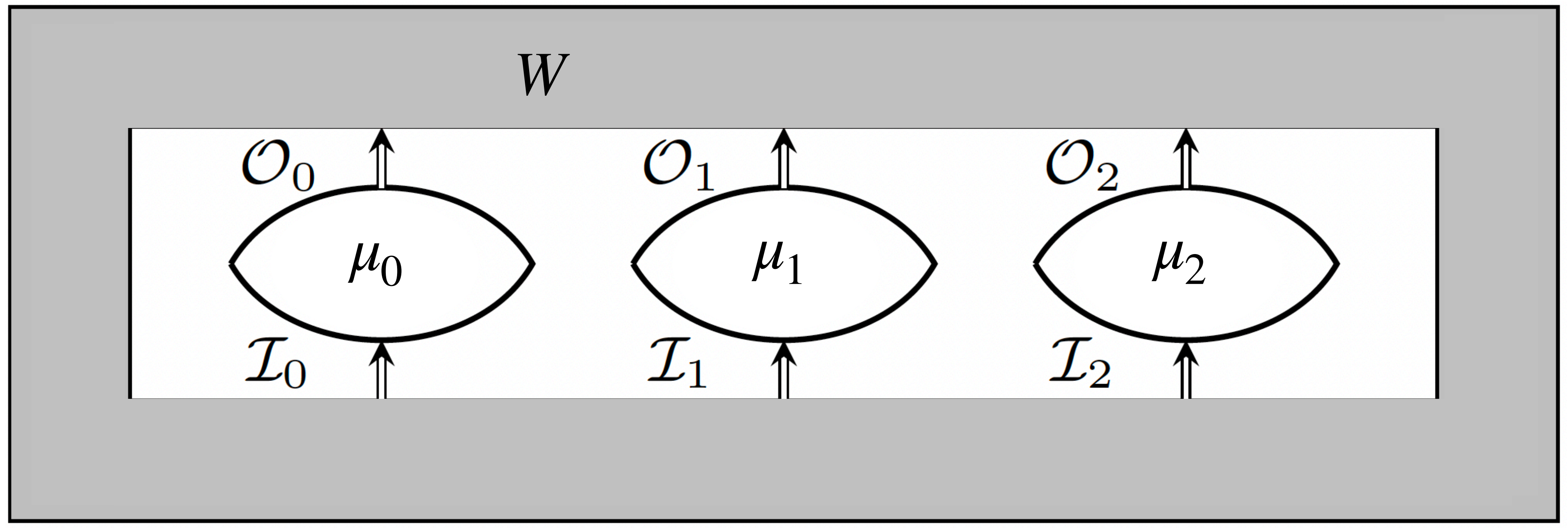
Process Matrix Formalism

Quantum Process W

$$\cdot\ W \in \mathscr{L}(\otimes_k I_k \otimes O_k) \quad \text{positive semi-definite}$$

$$\cdot\ \forall \{\mu_k \in CPTP(I_k, O_k)\}_k : Tr[W(\otimes_k \rho^{\mu_k})] = 1$$

$$\rho_{k|Pa(k)} : Choi(CPTP(O_{Pa(k)}, I_k))$$

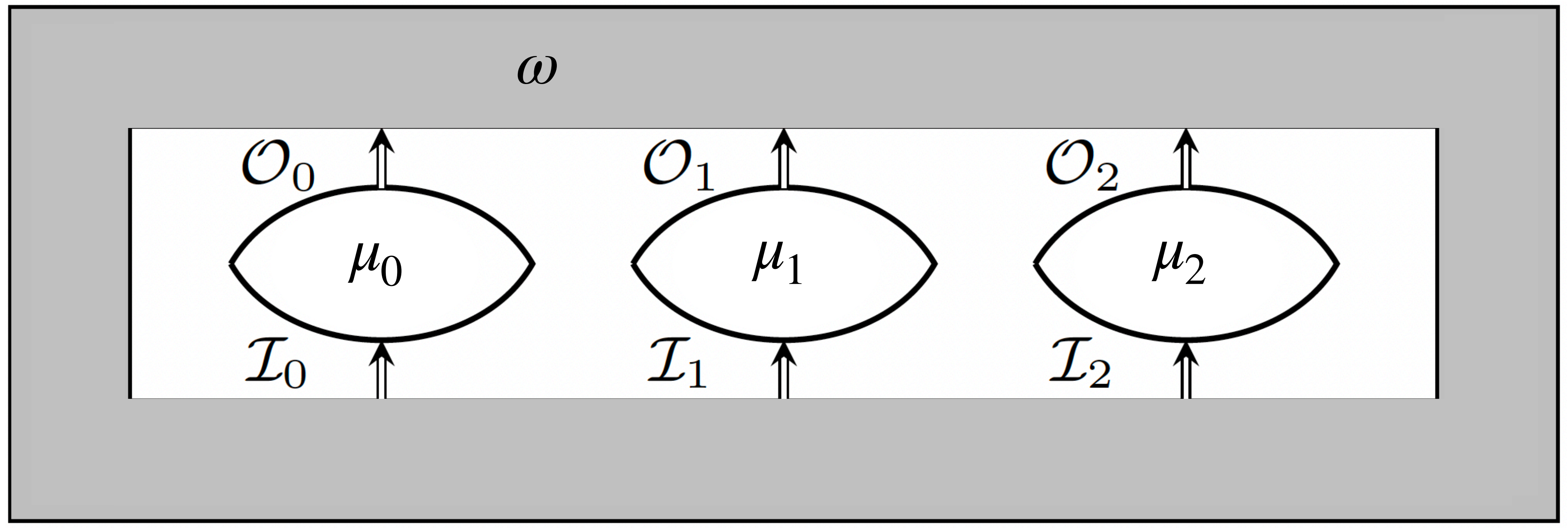


O. Oreshkov, F. Costa, Č. Brukner, Nature Comm. 3 (2012)

Classical-Deterministic Process ω

$\cdot\ \omega : \times_k O_k \rightarrow \times_k I_k$

$\cdot\ \forall \{\mu_k :\ I_k \rightarrow O_k\}\, \exists r \in \times_k I_k : \omega(\mu(r)) = r$



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Split-Node Causal Models

A split node causal model consist of:

· *a causal structure (directed ~~acyclic~~ graph, $G = (V, E)$)*

where each node v_i is a party with an input and output space

· *model parameters $\{\rho_{k|Pa(k)}\}$*

Barrett, J., Lorenz, R. & Oreshkov, O. Cyclic quantum causal models. Nat Commun 12, 885 (2021)

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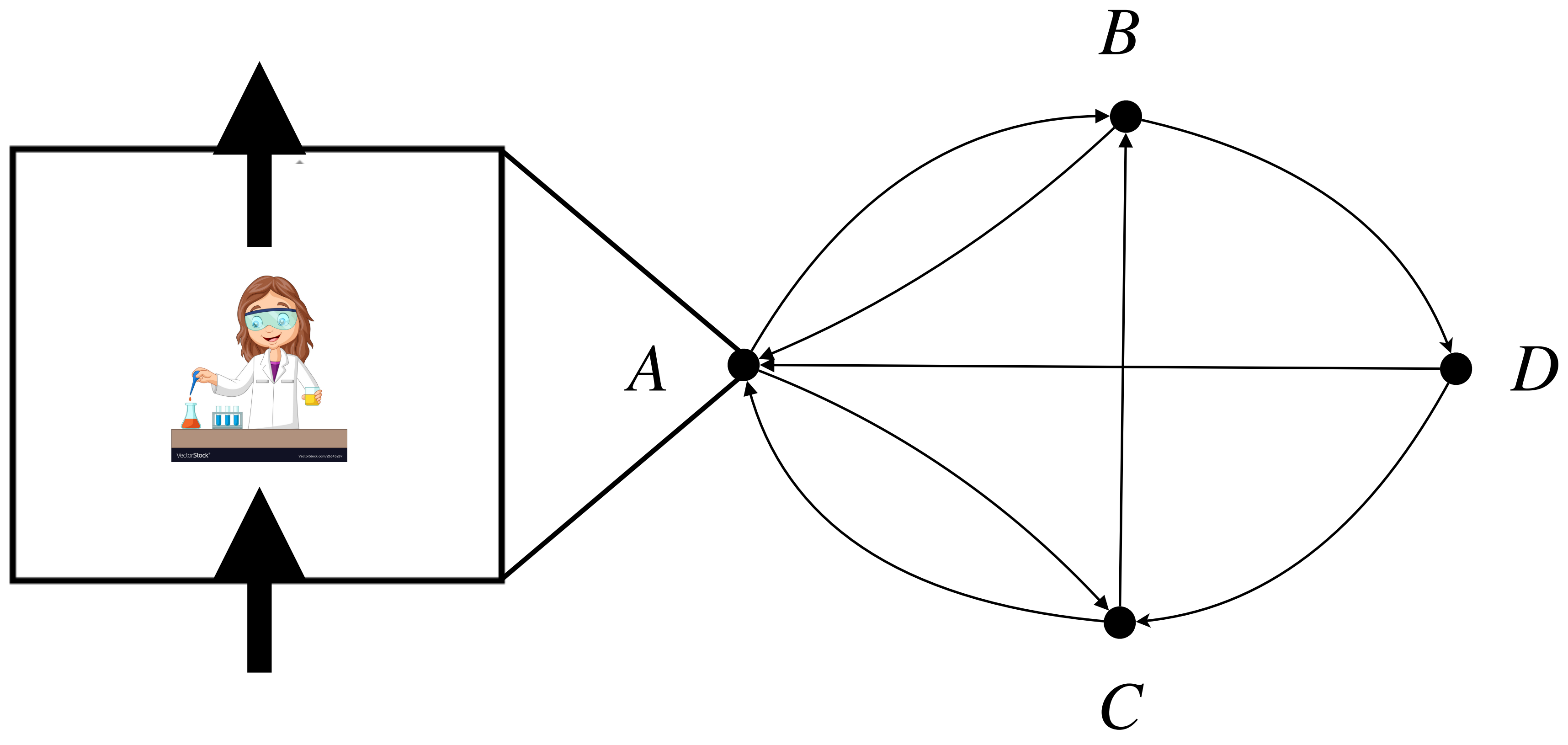

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Classical-Deterministic	Quantum
$\cdot\ \rho_{k\|Pa(k)} : O_{Pa(k)} \to I_k$	$\cdot\ \rho_{k\|Pa(k)} : Choi(CPTP(O_{Pa(k)}, I_k))$
$\cdot\ \omega := (\rho_{k\|Pa(k)})_k$	$\cdot\ W := \Pi_k\ \rho_{k\|Pa(k)}$

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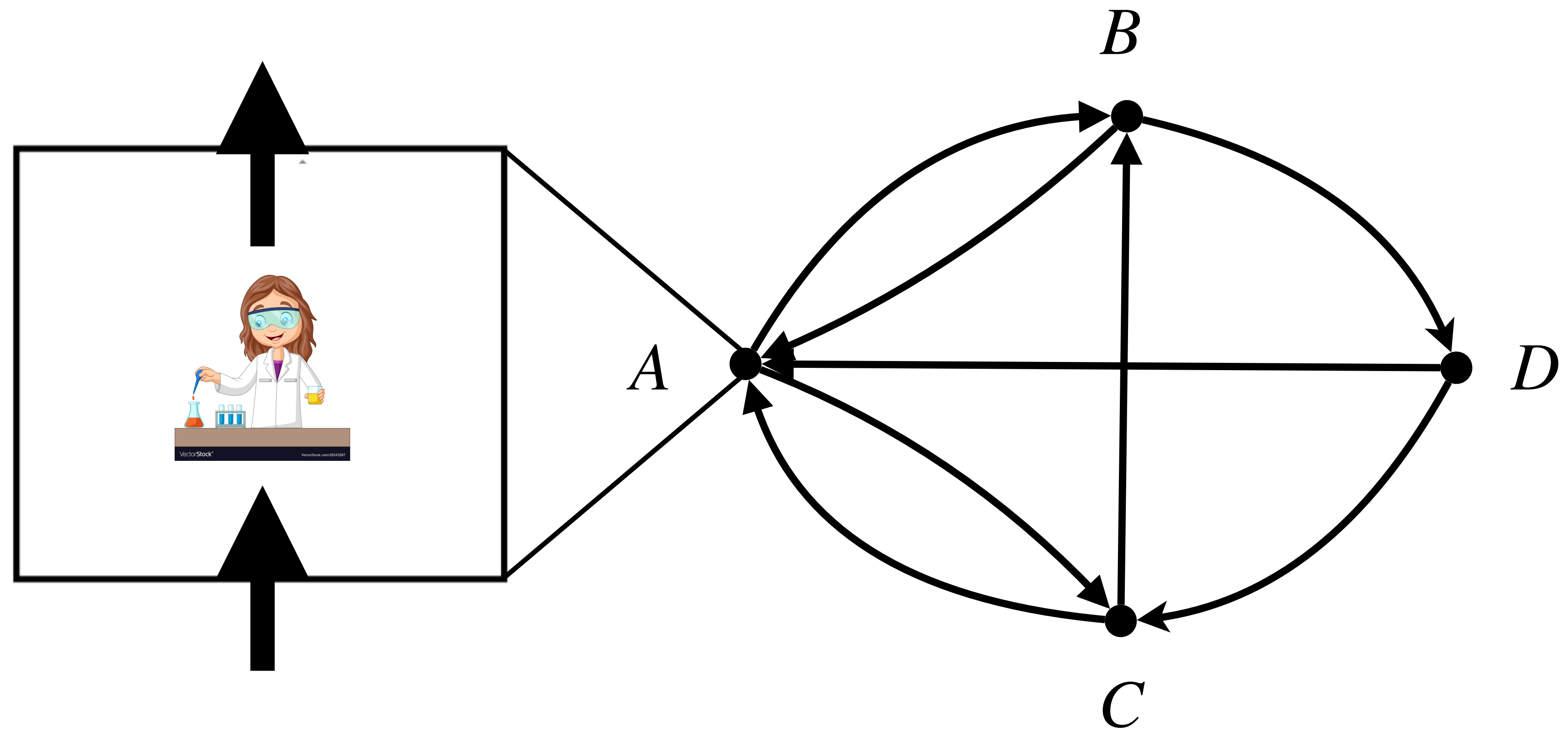

- $\rho_{A|B,C,D} : O_B \times O_C \times O_D \rightarrow I_A$
- $\rho_{B|A,C}$
- $\rho_{C|D}$
- $\rho_{D|B}$
- $\omega = (\rho_{A|B,C,D}, \rho_{B|A,C}, \rho_{C|A,D}, \rho_{D|B})$

Classical-Deterministic

- $\rho_{k|Pa(k)} : O_{Pa(k)} \rightarrow I_k$
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Quantum

- $\rho_{k|Pa(k)} : Choi(CPTP(O_{Pa(k)}, I_k))$
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Split-Node Causal Models

Important properties

<u>Faithfulness</u>*: iff every channel* $\rho_{A_k|Pa(A_k)}$ *is signalling from each* $A_i \in Pa(A_k)$ *to* A_k

<u>Consistency</u>*: If its a quantum/classical-deterministic process*

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Definition (Admissible causal structure): A causal structure $G = (V, E)$ *is* ***admissible*** *if and only if there exists a* ***faithful*** *and* ***consistent*** *causal model with causal structure* $G = (V, E)$

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*Definition(Admissible causal structure): A causal structure $G = (V, E)$ is **<u>admissible</u>** if and only if there exists a **faithful** and **consistent** causal model with causal structure $G = (V, E)$*

·Which are the causal structures that are (in)admissible?

· Is there a causal structure that is admissible for the quantum case and inadmissible in the classical-deterministic case?

Barrett, J., Lorenz, R. & Oreshkov, O. Cyclic quantum causal models. Nat Commun 12, 885 (2021)

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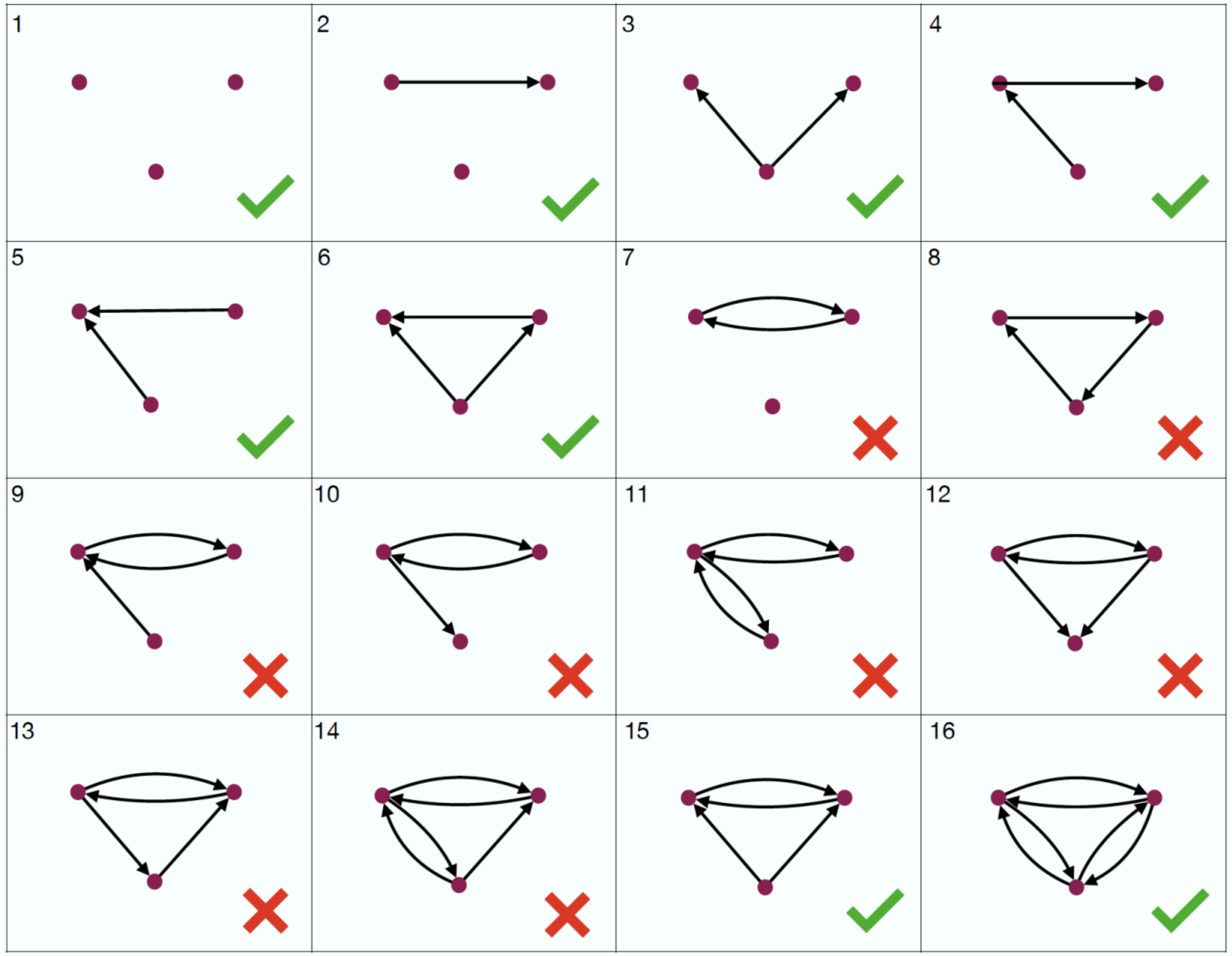
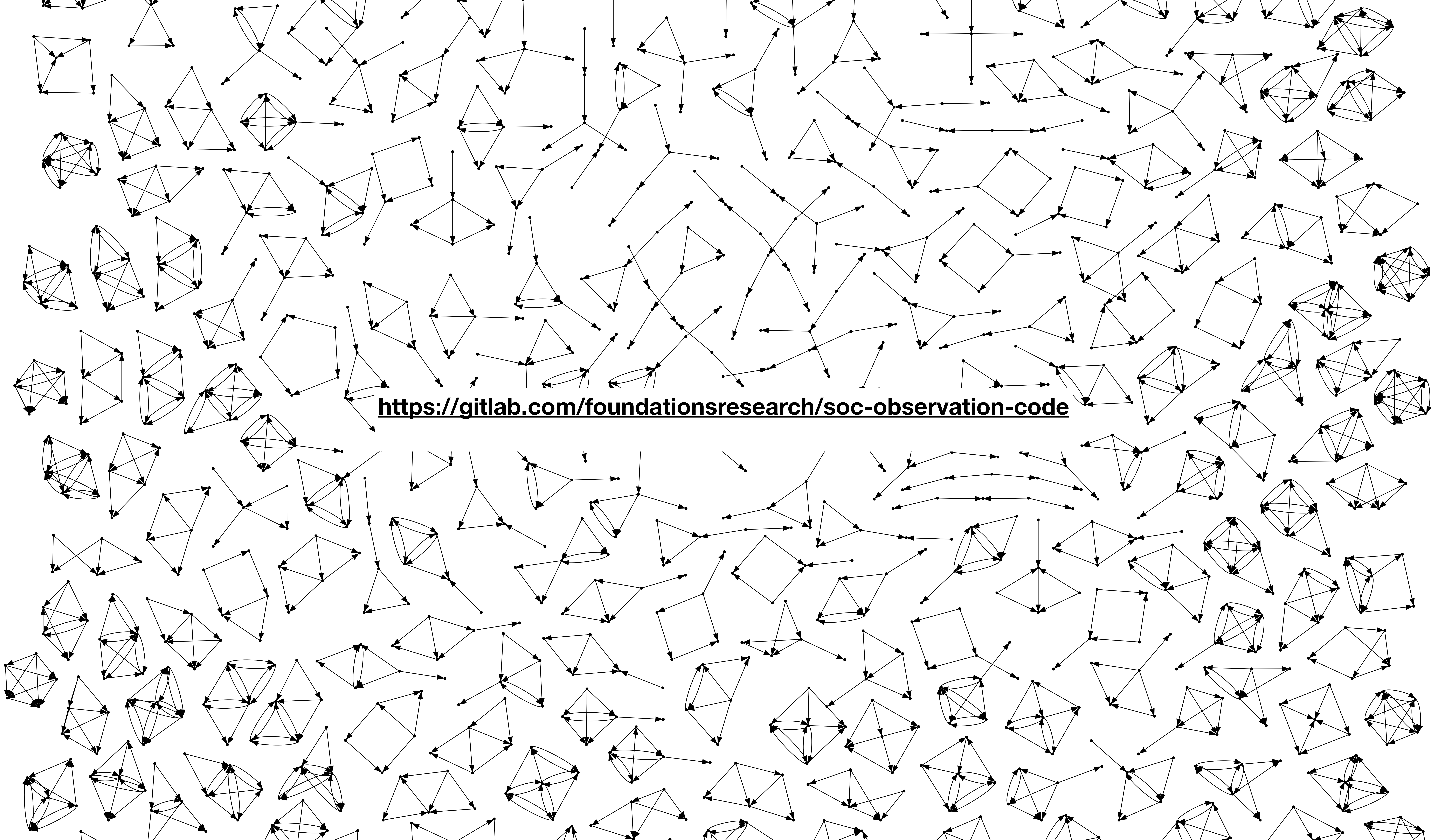
*Definition (Siblings-on-cycles graphs): A graph $G = (V, E)$ is a **siblings-on-cycles (SOC)** graph if and only if for all cycles in G there exists **siblings.***

*Definition (Siblings): Two nodes, A_i, A_j, are **siblings** if and only if they have at least one common parent $Pa(A_i) \cap Pa(A_j) \neq \varnothing$.*



Theorem *(Inadmissible causal structures): If a graph $G = (V, E)$ is not a SOC, then $G = (V, E)$ is inadmissible.*


https://gitlab.com/foundationsresearch/soc-observation-code

***Theorem**: If a graph $G = (V, E)$ is not a SOC, then $G = (V, E)$ is inadmissible.*

***Statement**: If $G = (V, E)$ is a SOC, then it is admissible.*

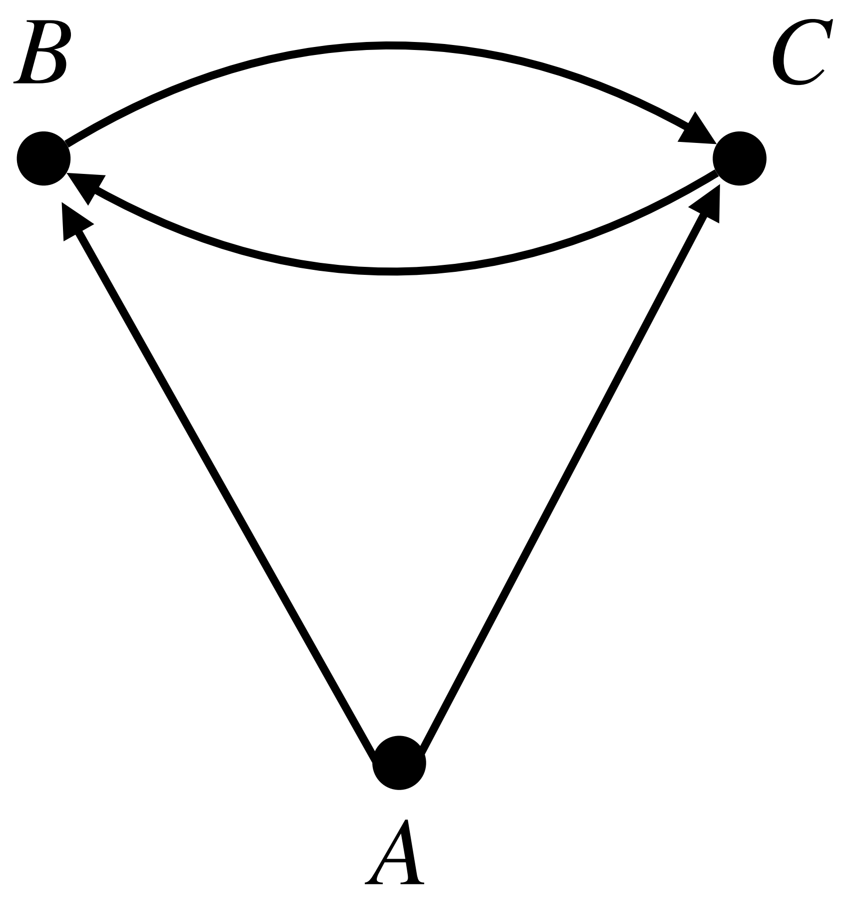
***Conjecture**: Given a SOC the following faithful classical-deterministic causal model is consistent:*

$\cdot\ I_k = \{0,1\}$

$\cdot\ O_k : Ch(k) \cup \{\ \perp\ \}$

$\cdot\ \rho_{k|Pa(k)} : O_{Pa(k)} \to I_k$

$$(o_j)_{j\in Pa(k)} \mapsto \Pi_{j\in Pa(k)}[o_j = k]$$



$\cdot\ \rho_{A|\varnothing} = const$

$\cdot\ \rho_{B|A,C} = [o_A = B] \cdot [o_C = B]$

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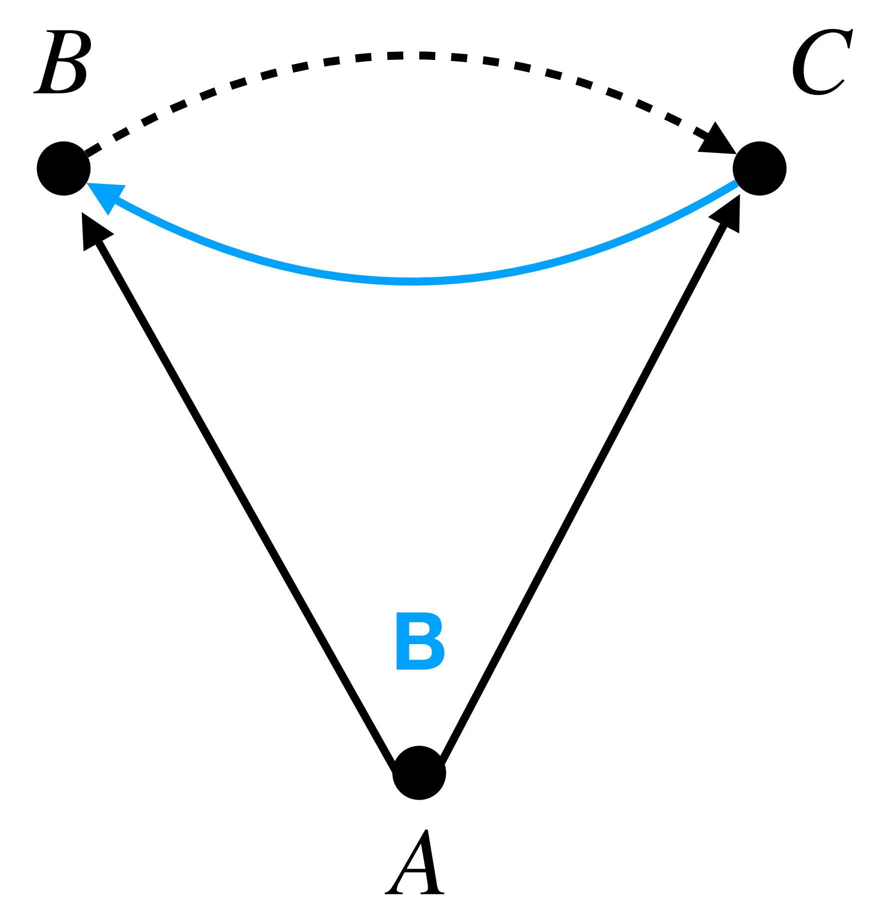

- $\rho_{A|\varnothing} = const$
- $\rho_{B|A,C} = \cancelto{1}{[o_A = B]} \cdot [o_C = B]$
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Admissible Causal Structures

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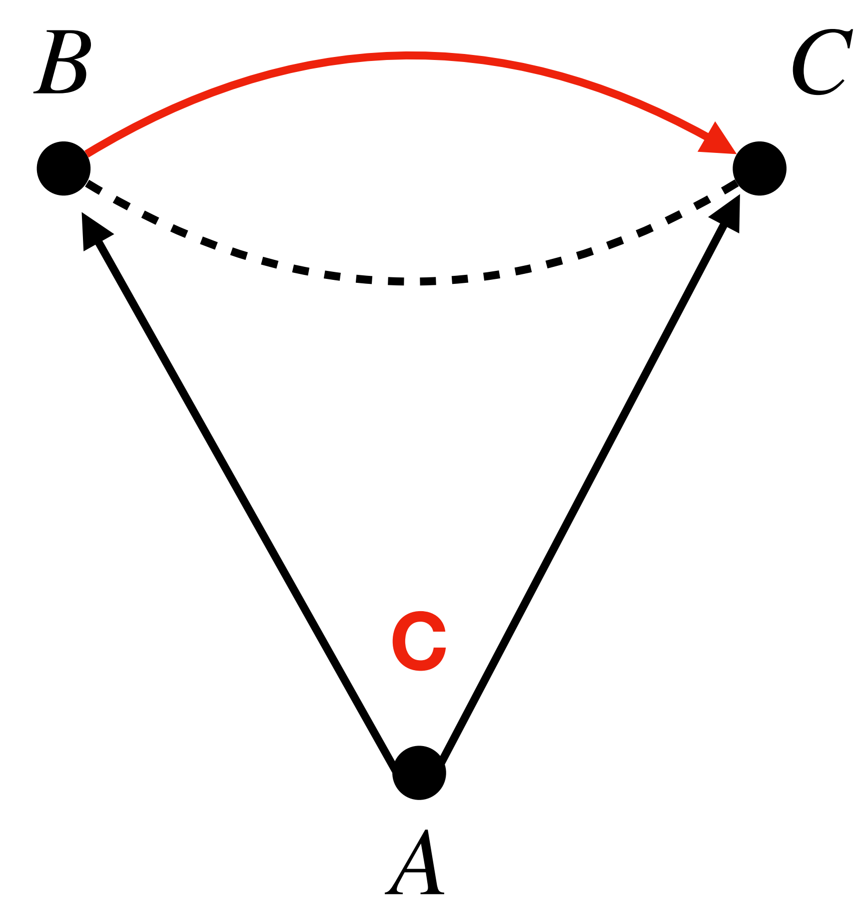

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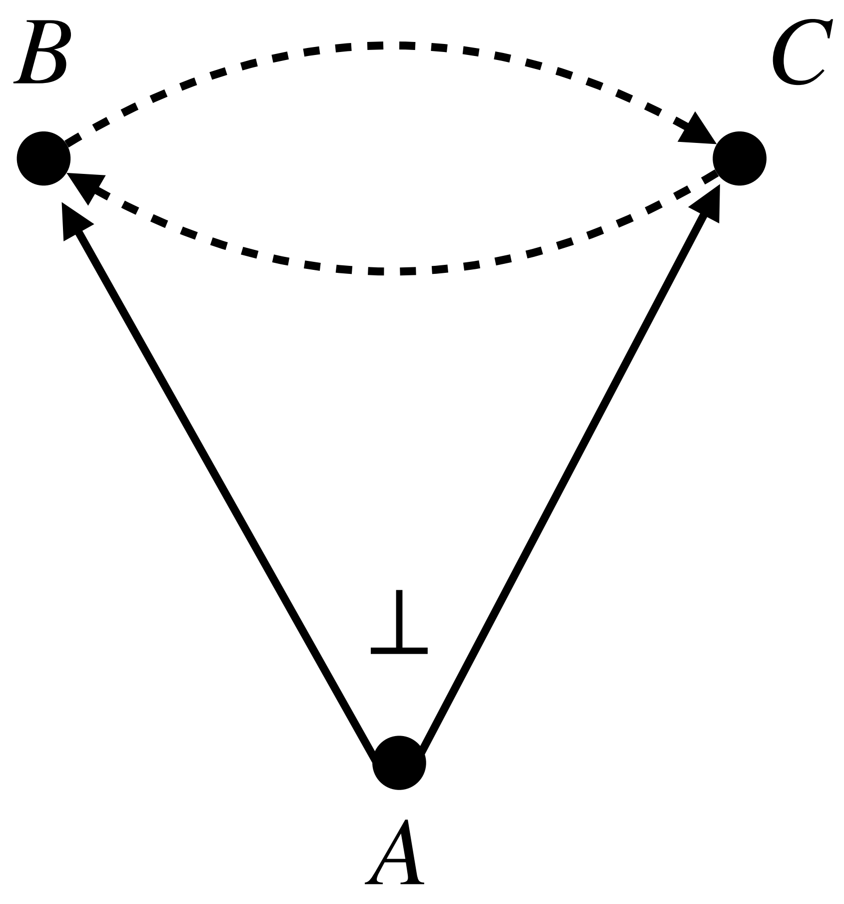

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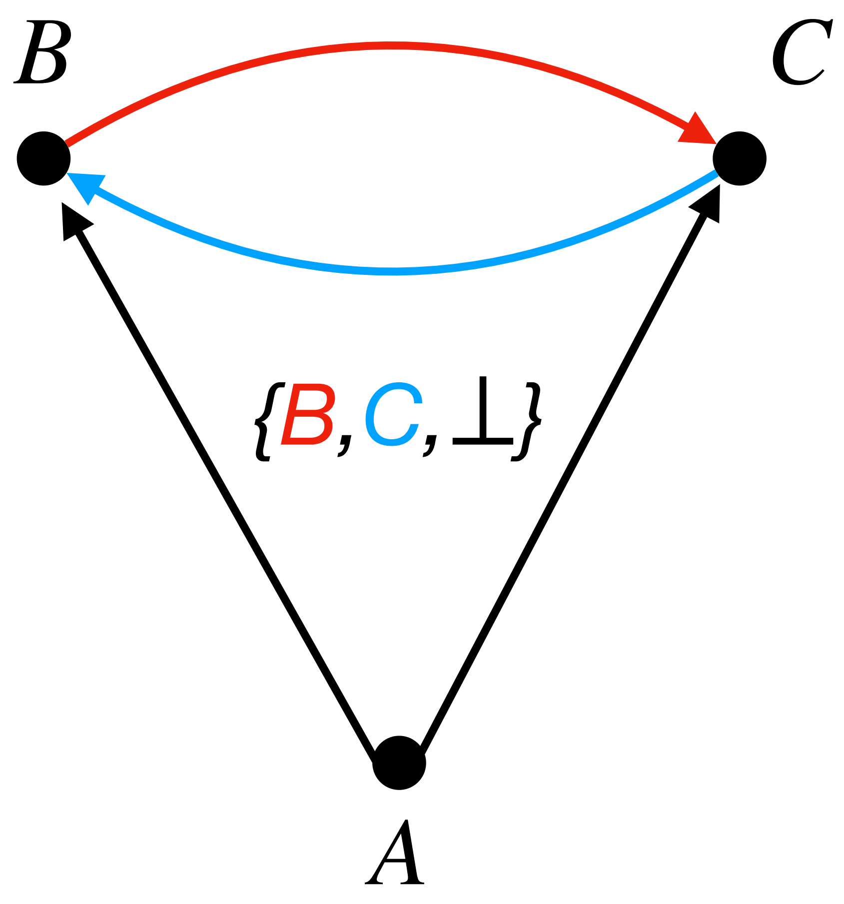
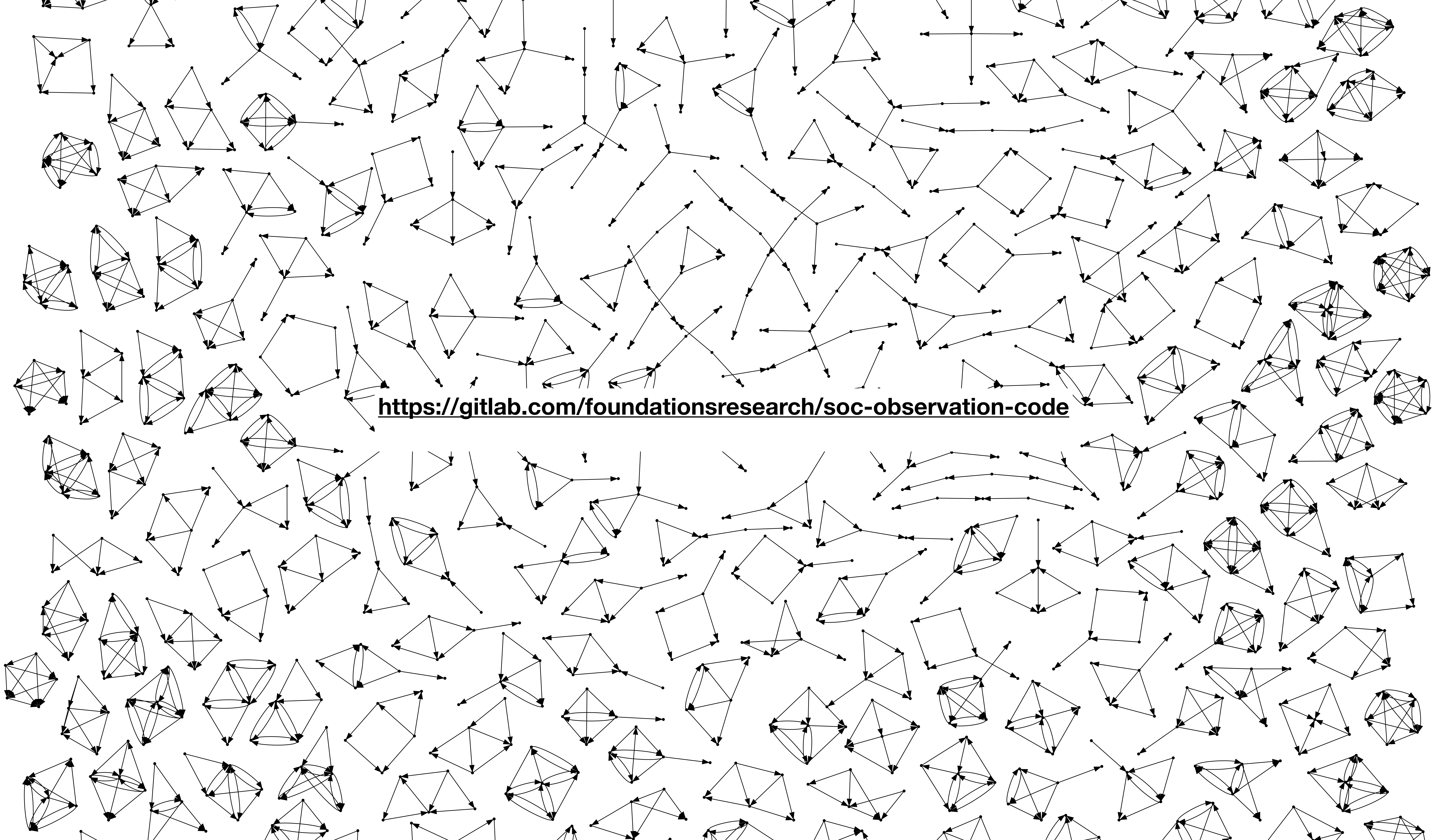
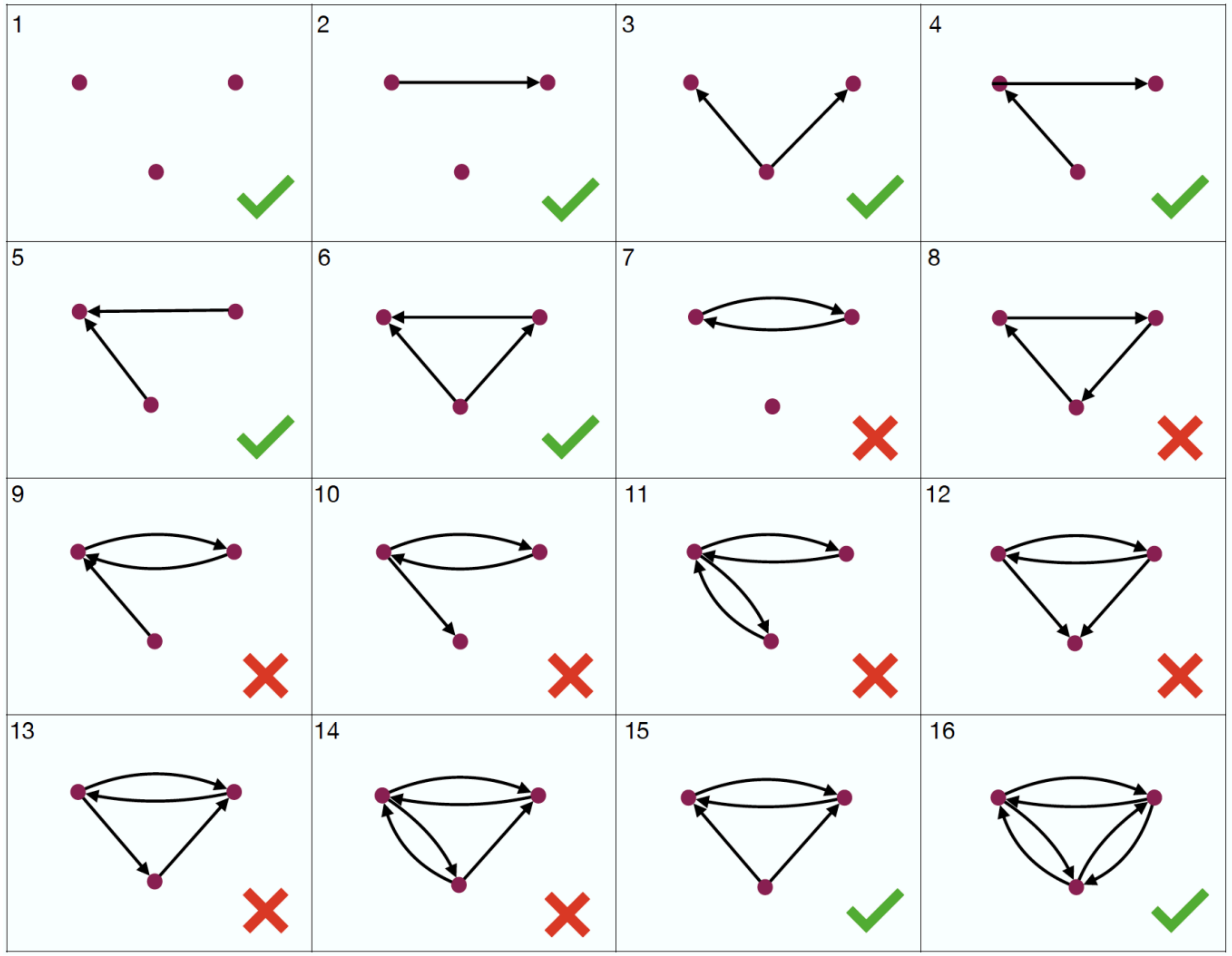



https://gitlab.com/foundationsresearch/soc-observation-code

If the conjecture holds:

Statement: *A graph $G = (V, E)$ is admissible if and only if its a SOC.*

Corollary: *The set of admissible causal structures in the quantum and in the classical-deterministic case coincide.*



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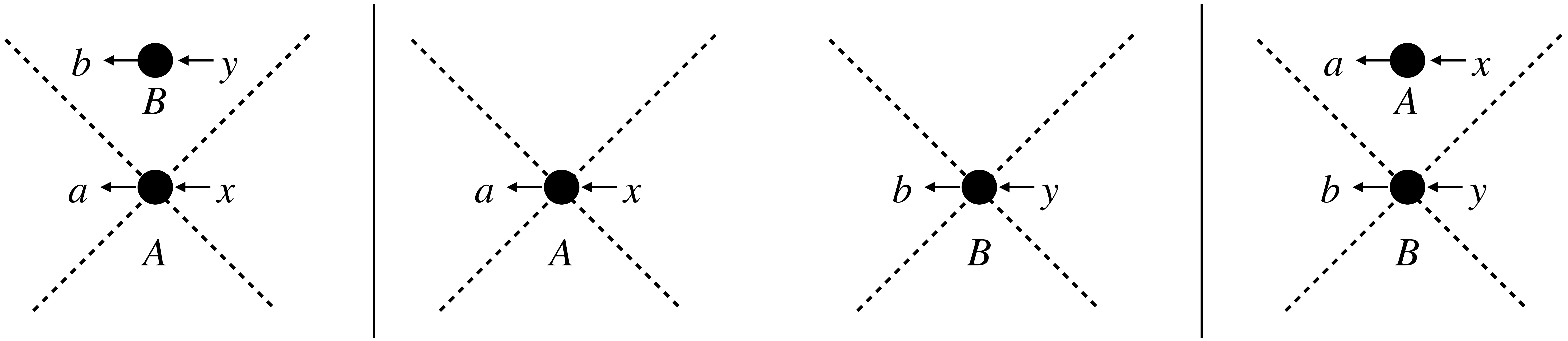
Correlations



Causal Correlations $$p(a, b \mid x, y) = q \cdot p(a \mid x) \cdot p(b \mid x, y, a) + (q - 1) \cdot p(b \mid y) \cdot p(a \mid x, y, b)$$

O. Oreshkov, F. Costa, Č. Brukner, Nature Comm. 3 (2012)

Definition (Causal Correlations): For a set of parties V, the set of correlations $p(a_V \mid x_V)$ are termed causal if and only if the decompose as follows:

$$p(a_V \mid x_V) = \sum_k q_k \, p(a_k \mid x_k) \, p_{(a_k, x_k)}(a_{V \setminus \{k\}}, x_{V \setminus \{k\}})$$

where $p_{(a_k, x_k)}(a_{V \setminus k}, x_{V \setminus k})$ *is causal.*

O. Oreshkon, C. Giarmatzi, NJP 18 (2016) *A. Abbott, C. Giarmatzi, F. Costa, C. Branciard, PRA 94 (2016)*

Questions

- *Which causal structures always lead to **causal** correlations?*
- *Which causal structures can lead to **non-causal** correlations?*

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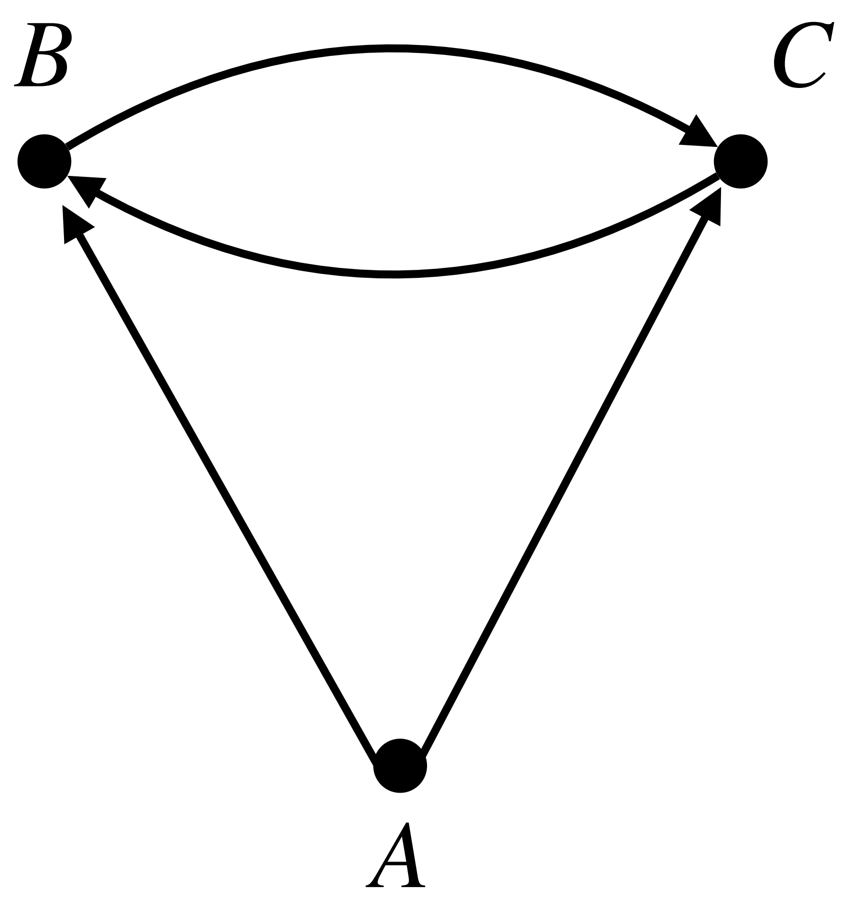
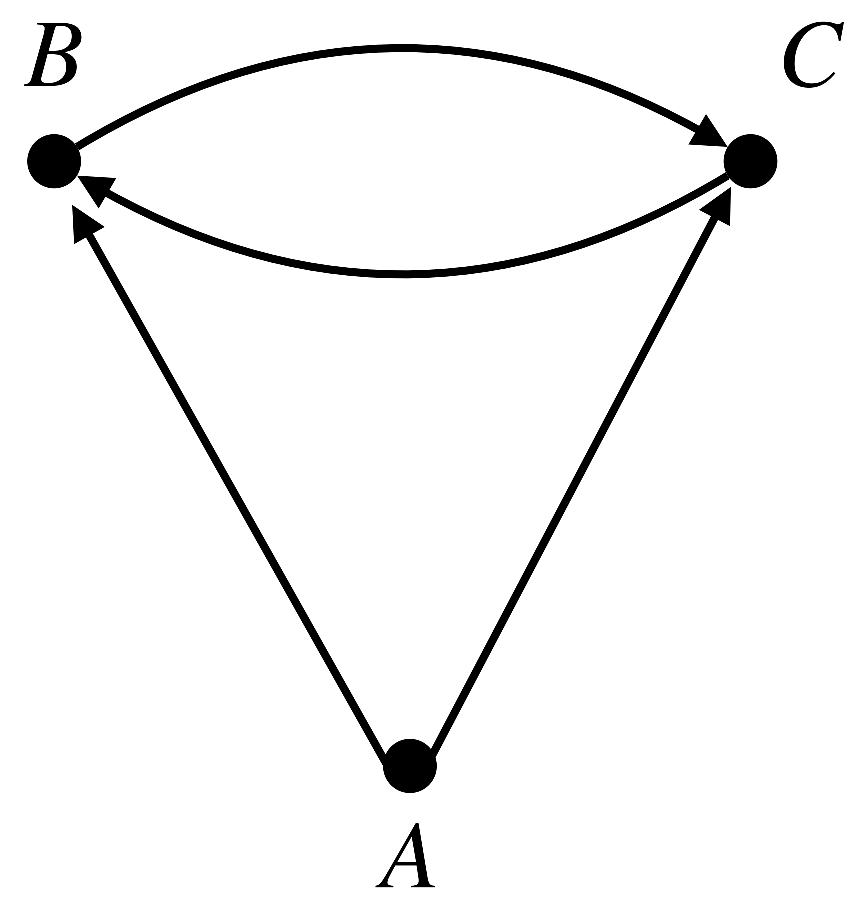
Definition *(Induced cycle):Let $G = (V, E)$ be graph. A cycle formed by some nodes $C \subseteq V$, is induced if the induced subgraph $G[C]$ is a cycle.*



Definition *(Induced cycle): Let $G = (V, E)$ be graph. A cycle formed by some nodes $C \subseteq V$, is induced if the induced subgraph $G[C]$ is a cycle.*

Theorem *(Causal Correlations): Let ω be a classical-deterministic process with causal structure $G = (V, E)$. If all cycles are induced then ω yields causal correlations.*



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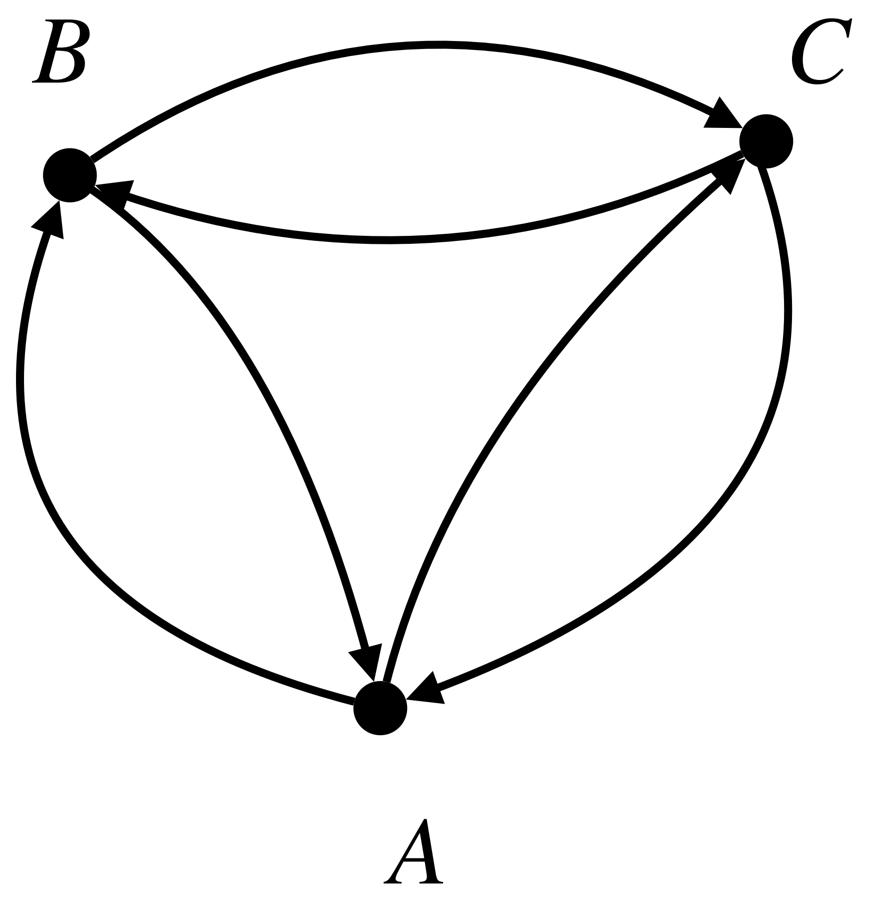
Theorem *(Non-causal Correlations): Let ω be a classical-deterministic process with causal structure $G = (V, E)$. If there exists a cycle where all the common parents are inside the cycle, i.e.,*

$$\cup_{i \neq j \in C} Pa(i) \cap Pa(j) \subseteq C$$

then ω produces non-causal correlations.



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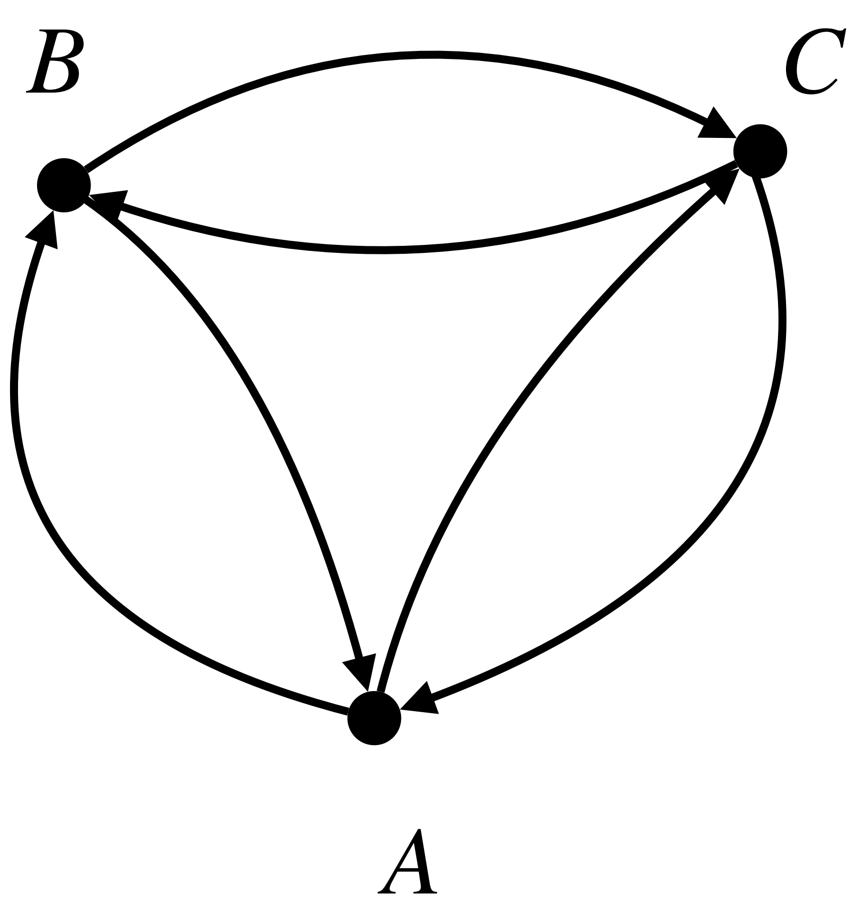
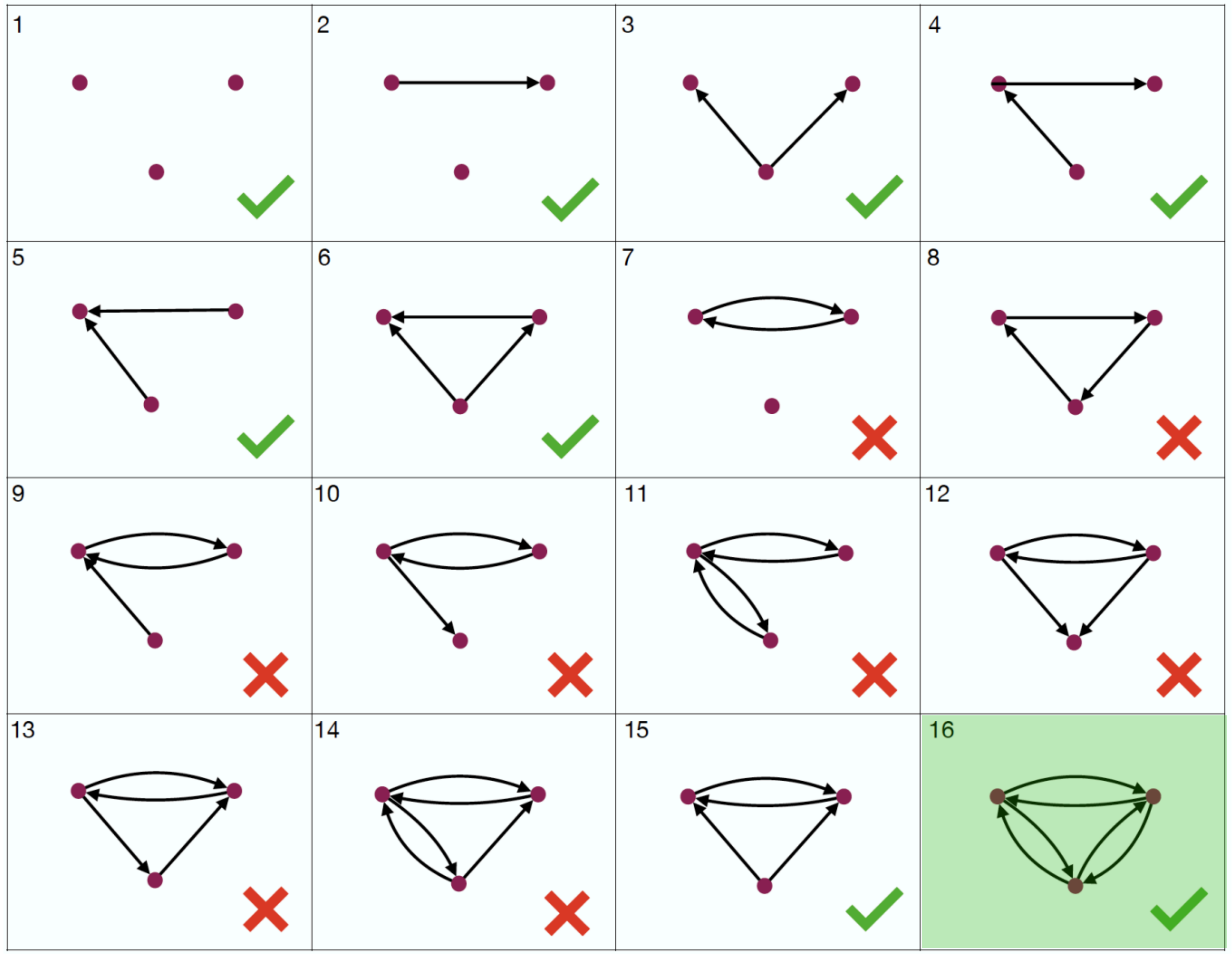
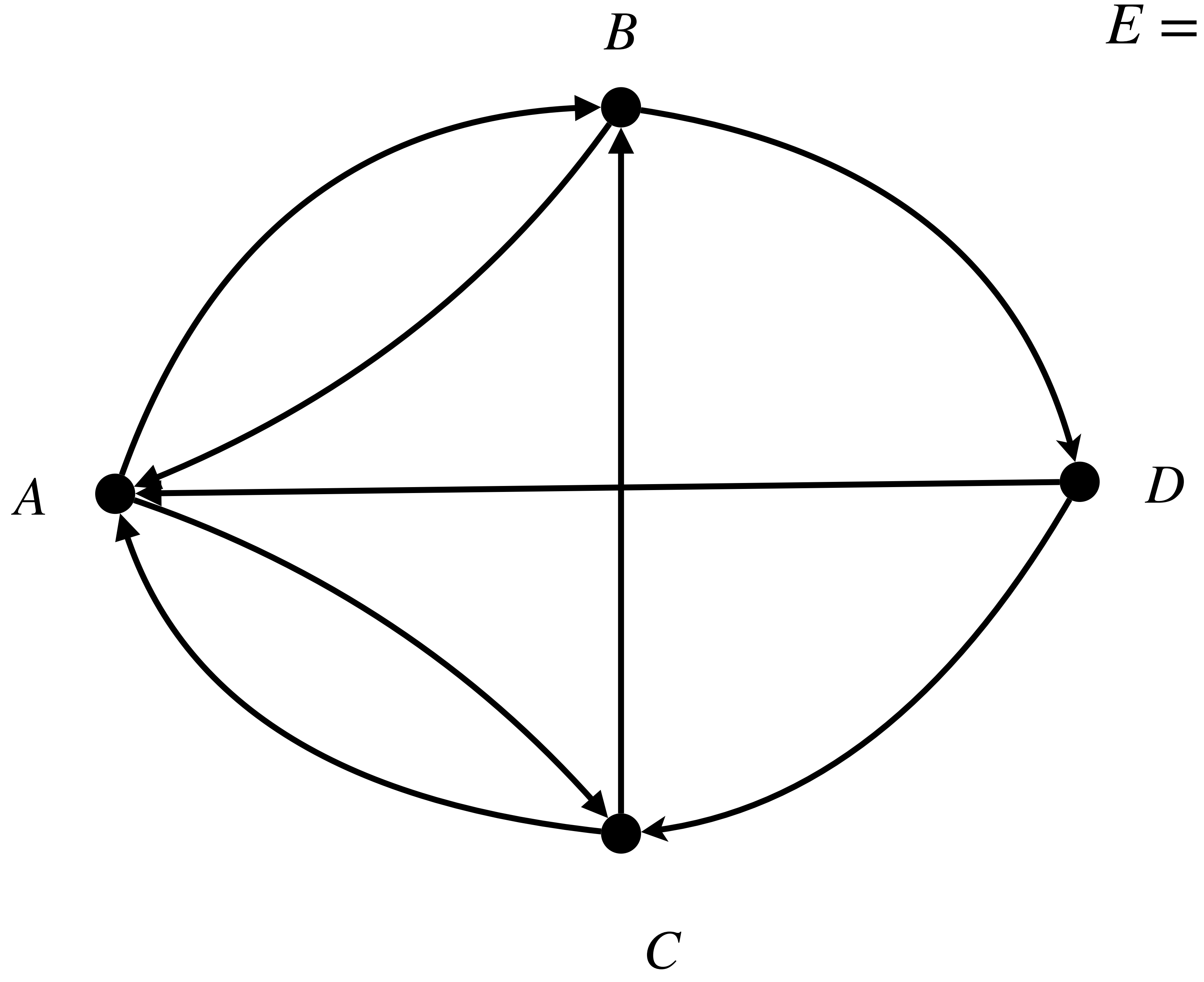




Summary

·Admissible causal structures



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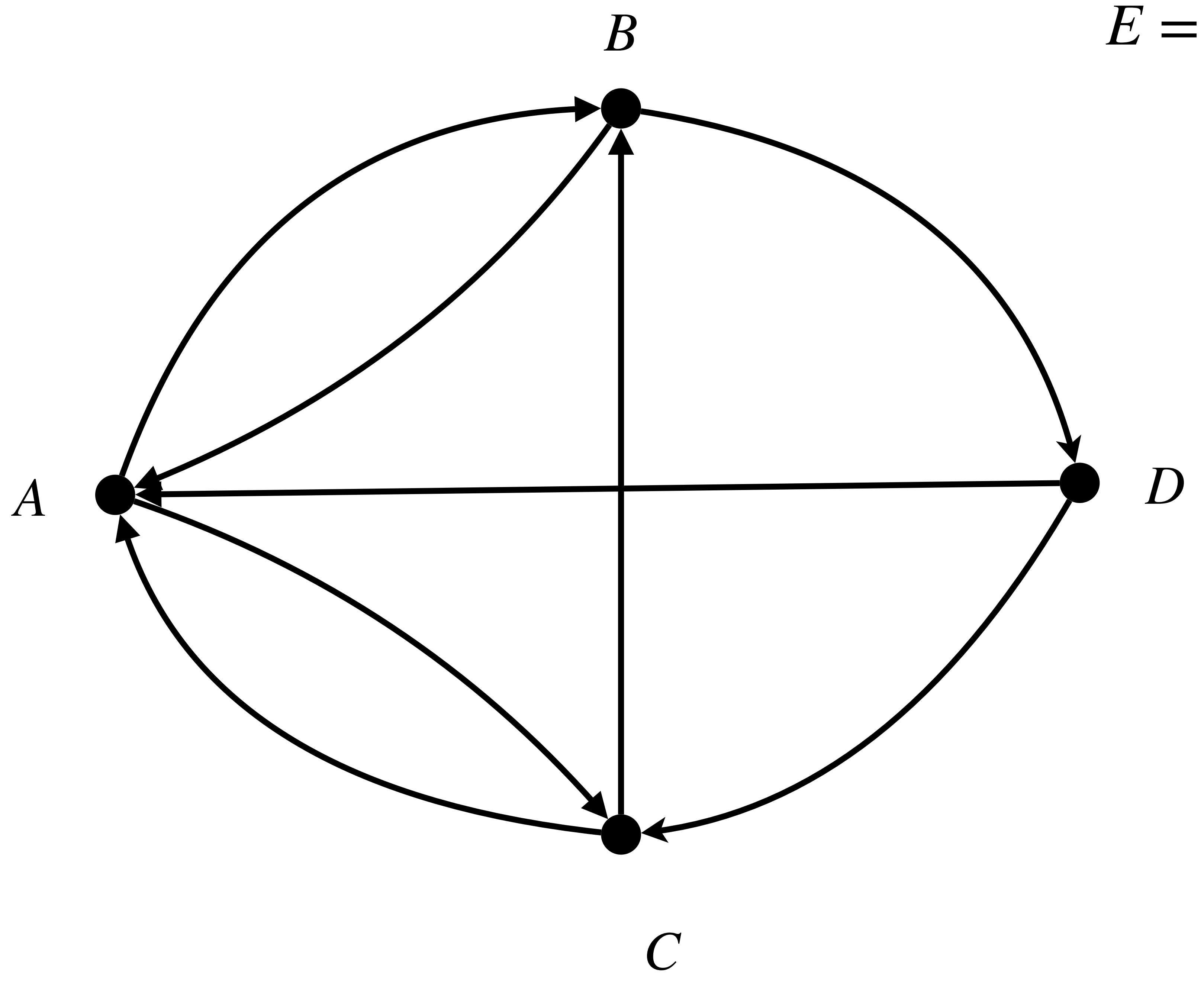
$$Cycles = \{(AB), (AC), (BDC), (ABDC)\}$$

Summary

- *Admissible causal structures*
- *Correlations*
    - *Causal*



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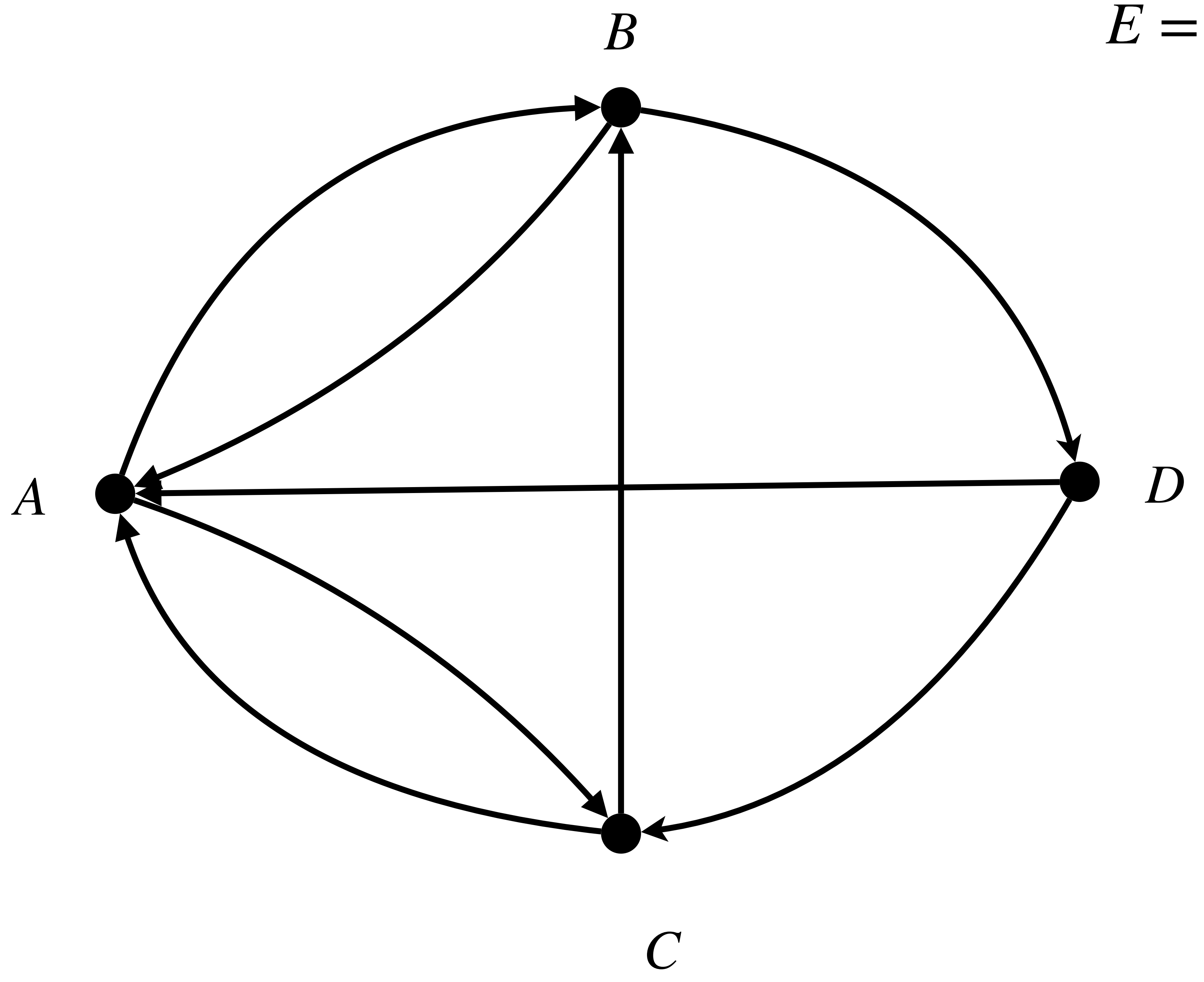

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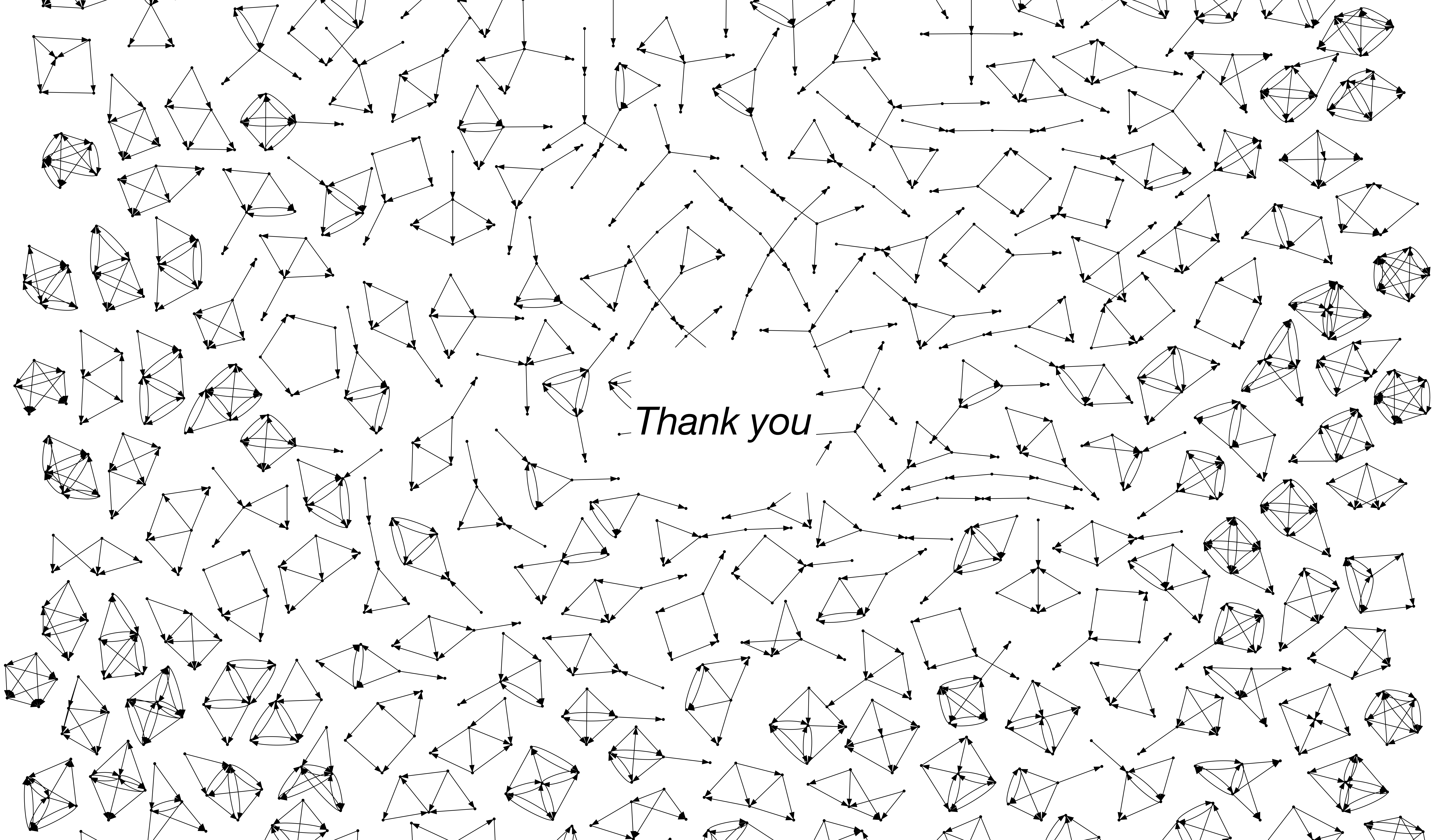

Thank you

https://gitlab.com/foundationsresearch/soc-observation-code

There is the code that:
1. produces the SOC graphs,
2. Checks the admissibility of them,
3. to print them

What about the the consistency of the model parameters?

Its in there with alpha and is checked up to 6

Define recursive function is OK, we do not search the whole space of possible solutions (eq.12)
Single core 31 hours verify the admissibility (that it is a process) 2 minutes to generate the graphs

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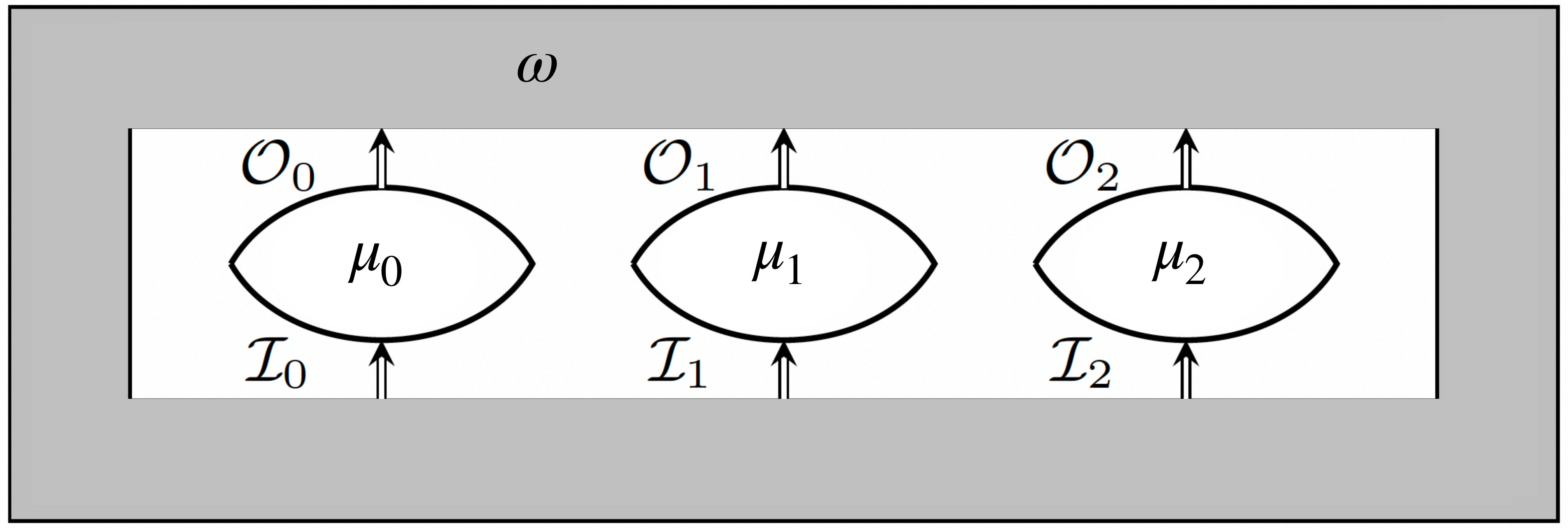
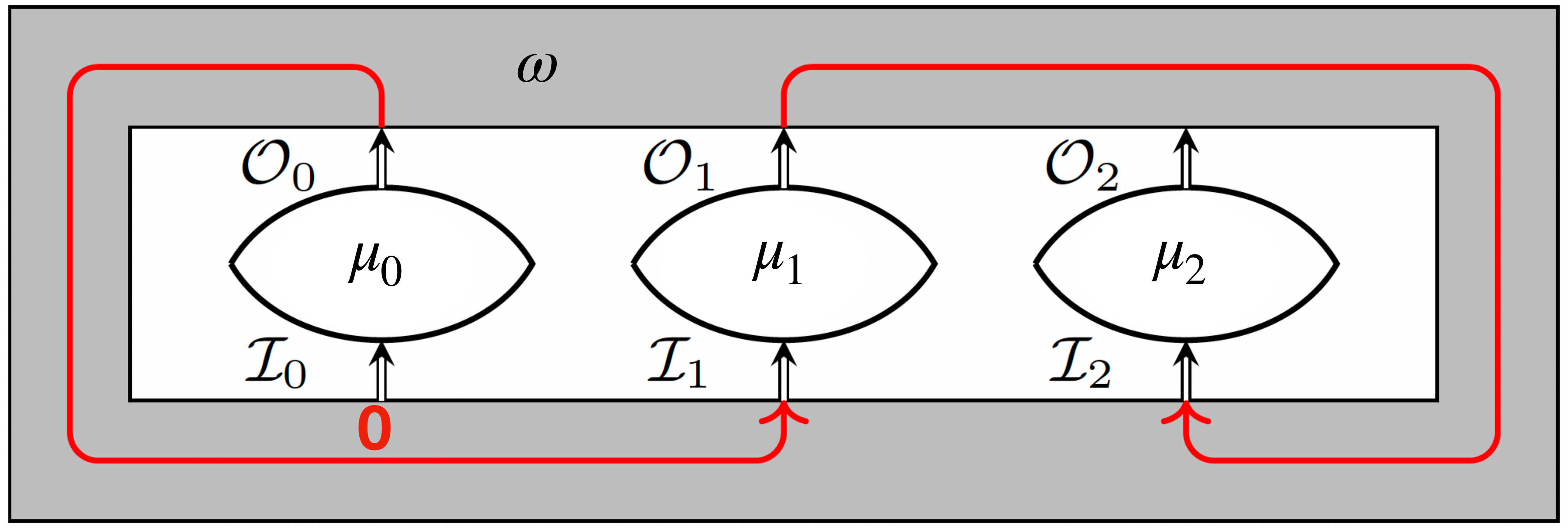


Ä. Baumeler, S. Wolf, NJP 18 (2016)
E.T., Ä. Baumeler, EPTCS 340, 2021, pp. 1-12

Classical-Deterministic Process ω

$$\cdot\ \omega : \{0,1\}^3 \to \{0,1\}^3$$

$$\cdot\ (o_0, o_1, o_2) \mapsto (0, o_0, o_1)$$



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