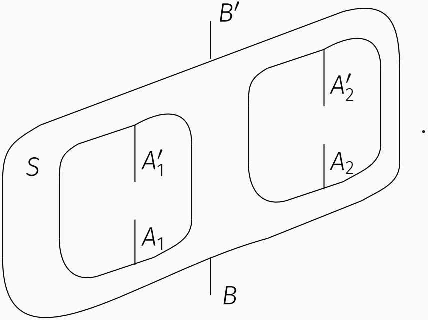
POLYCATEGORIES OF SUPERMAPS ON MONOIDAL CATEGORIES

Matt Wilson[1,2] Giulio Chiribella[2,3,4,5]

July 17, 2023

1. Department of Computer Science, University College London
2. Department of Computer Science, University of Oxford
3. HKU-Oxford Joint Laboratory for Quantum Information and Computation
4. QICI Quantum Information and Computation Initiative, Department of Computer Science, Department of Computer Science, The University of Hong Kong
5. Perimeter Institute for Theoretical Physics

Supermaps: boxes with holes



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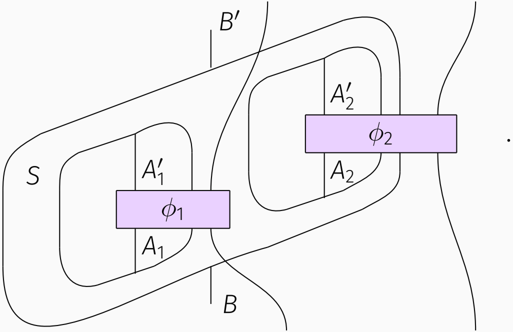
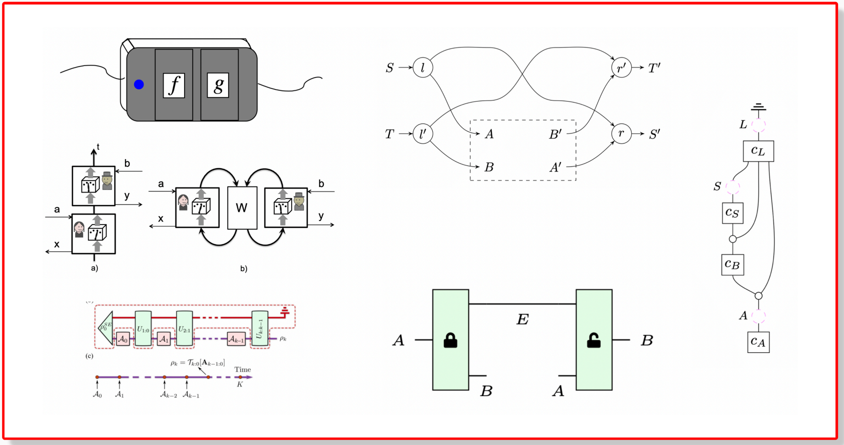
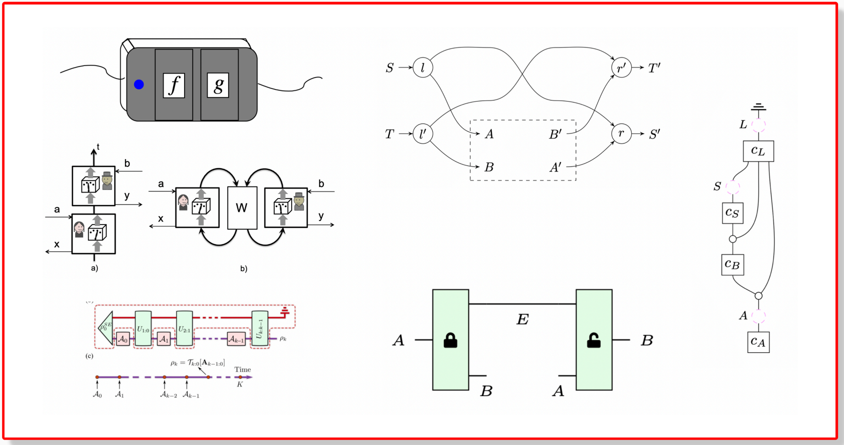




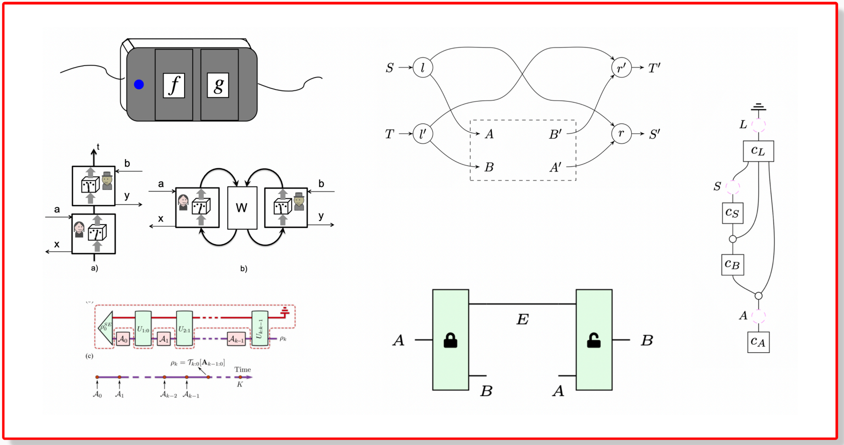
[G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, arXiv 2009, PRA 2013]

Supermaps: Black box Holes



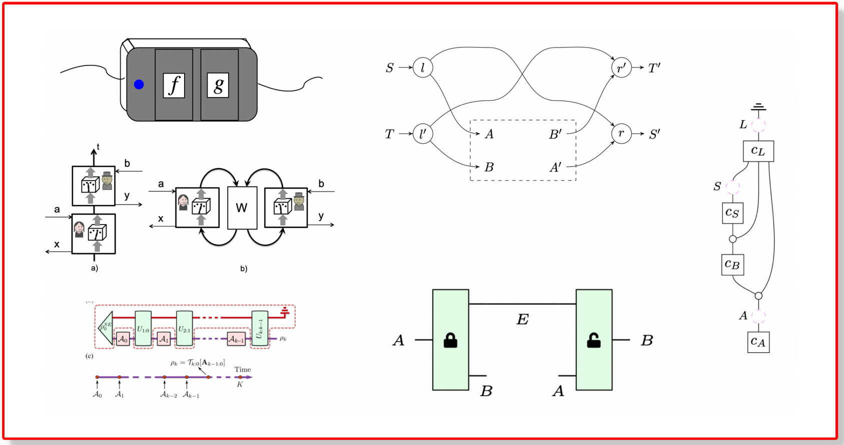
[O. Oreshkov, F. Costa, and Č. Brukner, Nat Comms 2012]

Supermaps: Black box Holes

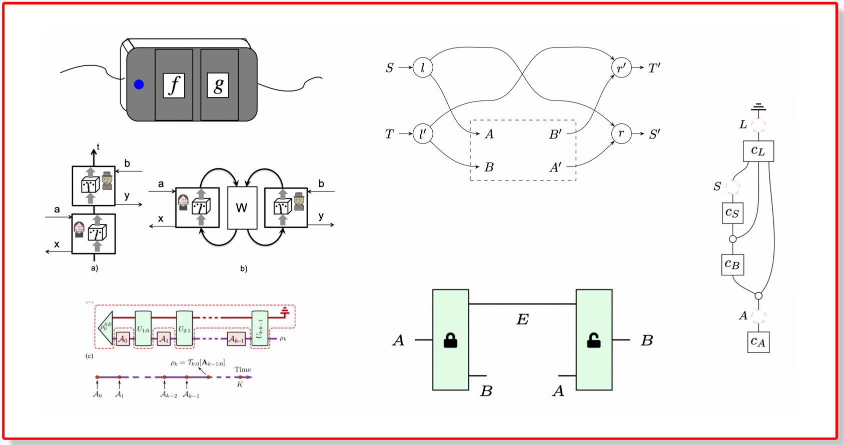


[F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, PRA 2015]

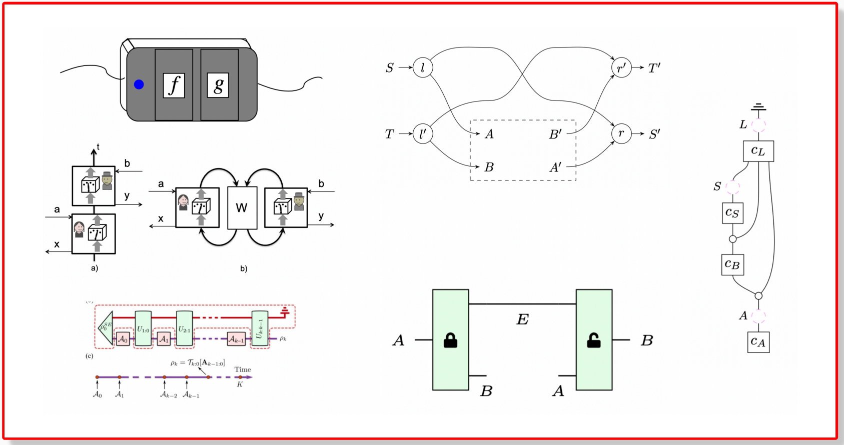
SUPERMAPS: BLACK BOX HOLES



[M. Riley, arXiv 2018]



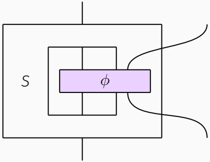
[F. Genovese, F. Loregian, and D. Palombi, arXiv 2021]



[R. Lorenz and S. Tull, arXiv 2023]

How Are Supermaps Defined?

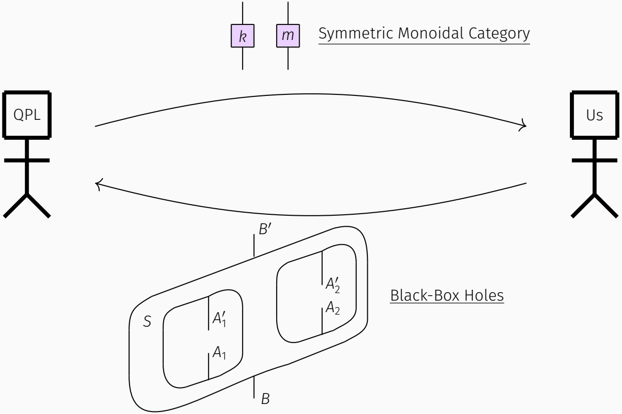
Aesthetic problems:



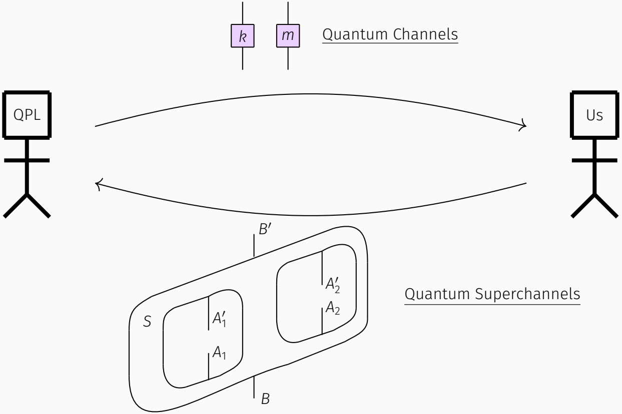
Technical Problems:

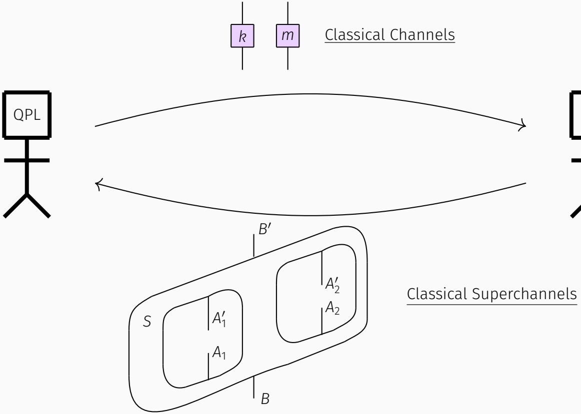
- General Hilbert spaces
- Supermaps on Operational Probabilistic Theories

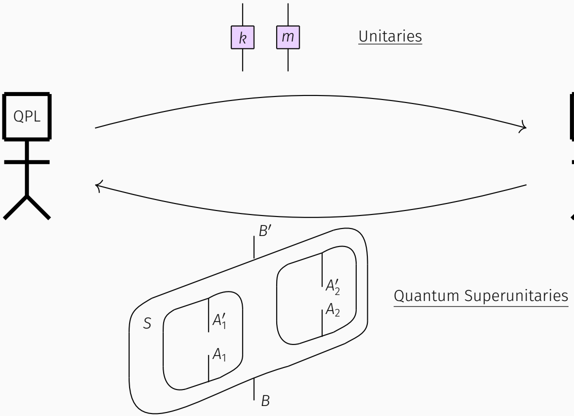
Categorical Supermaps P.1



Categorical Supermaps P.1

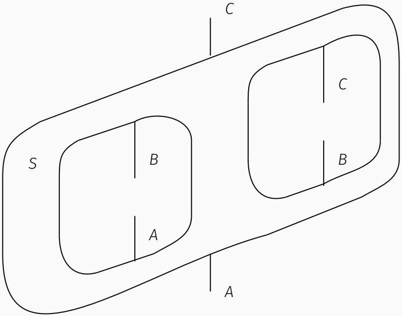



k
m
Classical Channels
QPL
Us
B′
S
A′1
A1
A′2
A2
B
Classical Superchannels


k
m
Unitaries
QPL
Us
B′
S
A′1
A1
A′2
A2
B
Quantum Superunitaries

Categorical Supermaps P.2

Polycategorical structure and enrichment



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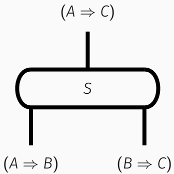

Categorical Supermaps P.2

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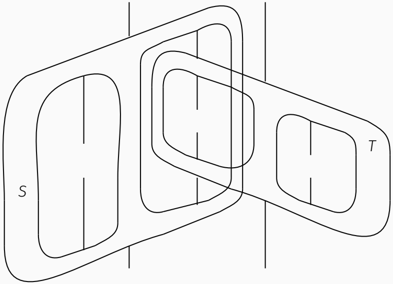

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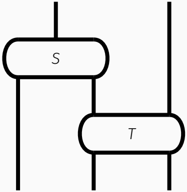
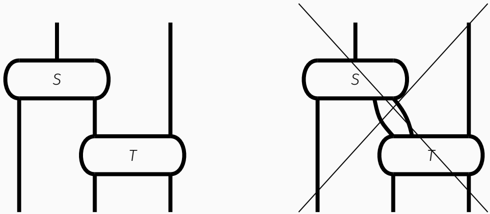


[M. Szabo, Comms in Algebra 1975]

Categorical Supermaps P.2

Polycategorical structure and enrichment



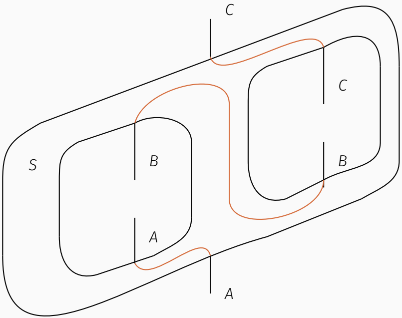
Examples

- **QS**: Quantum Superchannels
- **CS**: Classical Superchannels
- **Su**: Quantum Superunitaries

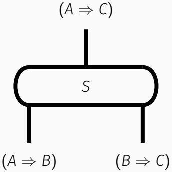
[A. Kissinger, S. Uijlen, LMiCS 2019]

Categorical Supermaps P.2

Polycategorical structure and **enrichment**



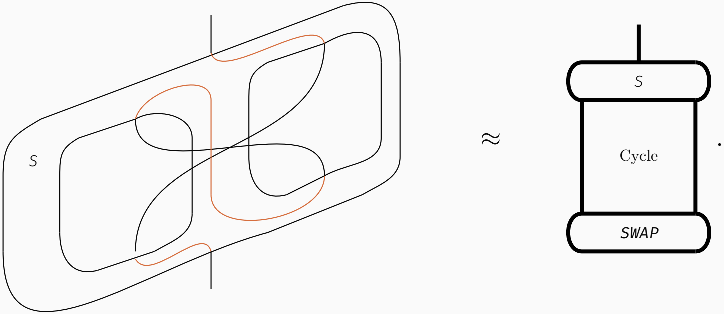
$\approx$



The polycategories **QS**, **CS**, **SU** enrich **QC**, **Stoch**, **U** respectively

[P. T. Johnstone, 1983][M. Wilson, G. Chiribella, arXiv 2022][M. Rennela and S. Staton, ENTCS 2018]

Polycategorical structure and enrichment



[D. Jia, and N. Sakharwade, PRA 2018][P. Guérin, M. Krumm, C. Budroni, and Č. Brukner, New Journal of Physics 2019]

Plan of Action

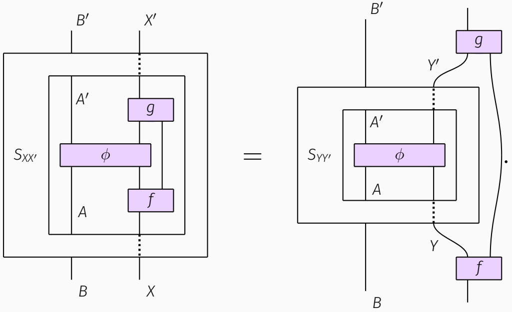
Categorical constructions for classes of supermaps:

- Higher Order Causal Categories
- Combs and Profunctor Optics
- **Supermaps by local applicability**

[A. Kissinger, S. Uijlen, LMiCS 2019] [W. Simmons, A. Kissinger, arXiv 2022] [B. Coecke, T. Fritz, and R. W. Spekken, Information and Computation 2016] [M. Román, ACT 2020] [J. Hefford C. Comfort, arXiv 2022] [G. Boisseau, C. Nester, M. Roman, 2022 arXiv] [M. Earnshaw, J. Hefford, M. Roman, arXiv 2023]

Locally Applicable Transformations

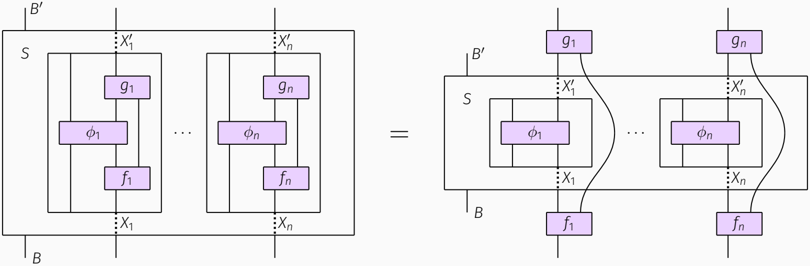
$$S : (A \Rightarrow A') \longrightarrow (B \Rightarrow B') \qquad S_{XX'} : \mathsf{C}(AX, A'X') \to \mathsf{C}(BX, B'X')$$



Locally Applicable Transformations

$$S : \times_{i=1}^{n}(A_i \Rightarrow A_i{}') \longrightarrow (B \Rightarrow B')$$

$$S^{X_1'\ldots X_n'}_{X_1\ldots X_n} : \times_{i=1}^{n}\mathsf{C}(A_iX_i, A_i^{'}X_i^{'}) \to \mathsf{C}(BX_1\ldots X_n, B^{'}X_1^{'}\ldots X_n^{'})$$



Where it Works … OPT-like Cases

Theorem

There is a one-to-one correspondence between locally applicable transformations on quantum channels and deterministic quantum supermaps of type

$$\times_{i=1}^{n}(A_i \Rightarrow A_i') \longrightarrow (B \Rightarrow B')$$

In categorical language, there are equivalences

$$\mathsf{QS} \cong_{\mathsf{Multi}} \mathsf{Lot}[\mathsf{QC}] \qquad \text{and} \qquad \mathsf{CS} \cong_{\mathsf{Multi}} \mathsf{Lot}[\mathsf{Stoch}]$$

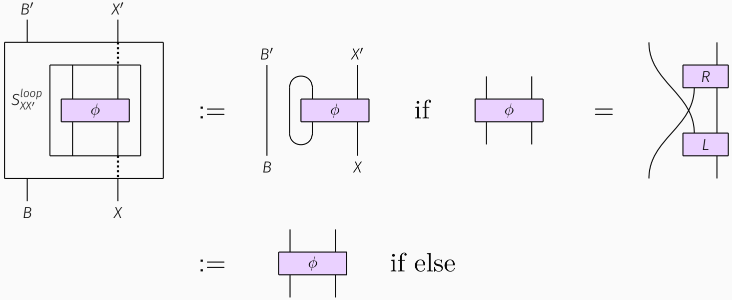
But: we can only say

$$\mathsf{Su} \subseteq_{\mathsf{Multi}} \mathsf{lot}[\mathsf{U}]$$

[M. Wilson, G. Chiribella, and A. Kissinger, arXiv 2022]

Sneaky Non-Linearity

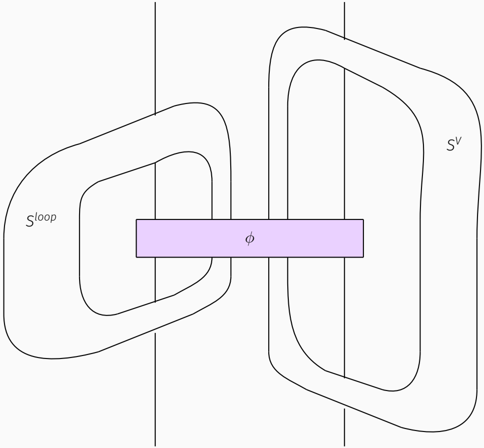
$$\mathsf{Su} \subseteq_{\mathsf{Multi}} \mathsf{lot}[\mathsf{U}]$$



[M. Wilson, G. Chiribella, arXiv 2022]

Double Trouble From the Loop Section

Parallel composition problem: same culprit



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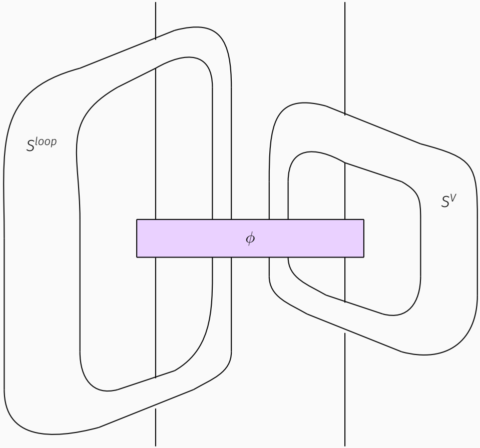

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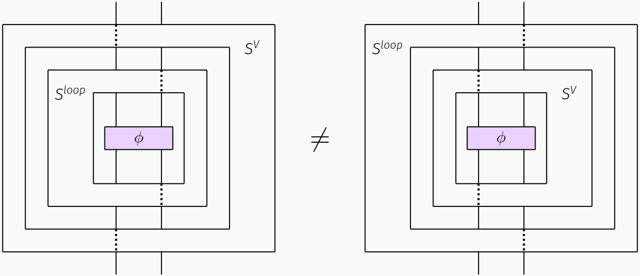
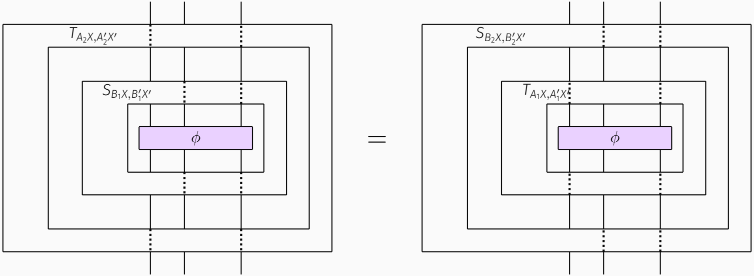


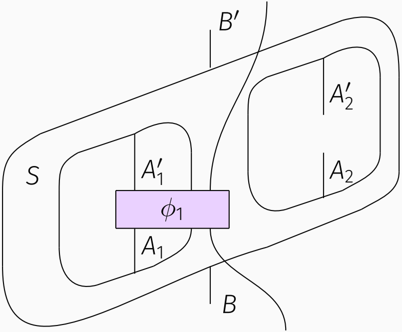
Strongly Locally Applicable Transformations

A slot S is a locally-applicable transformation such that for every locally applicable transformation T:



- Taking centre
- Taking a bicommutant

Valid supermaps can be defined inductively:



Polyslots (un-formal)

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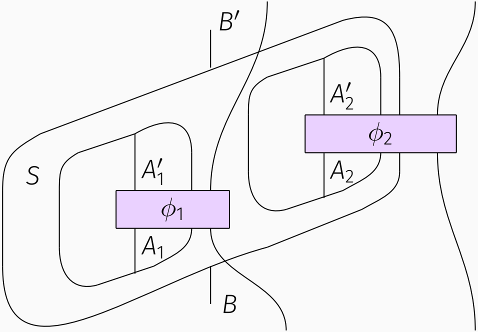

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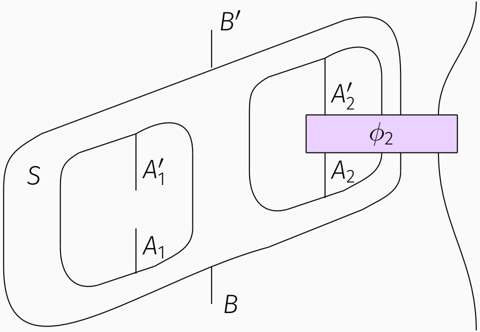

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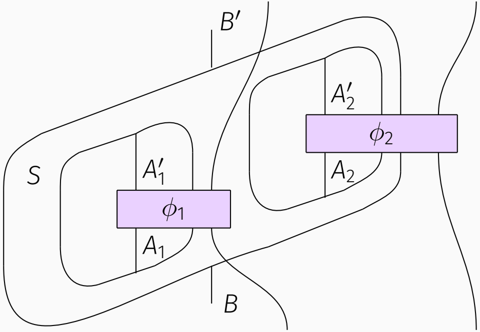
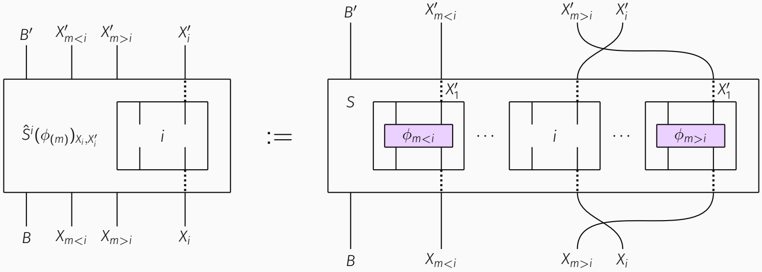


Polyslots (semi-formal)

$S : \times_{k=1}^{n}(A_k \Rightarrow A_k^{'}) \to (B \Rightarrow B')$ if for every i and every $\underline{\phi}_{1\ldots i-1}, \underline{\phi}_{i+1\ldots|\underline{\mathbf{A}}|}$ then the family of functions given by



Is a slot of type

$$S^i(\phi_{(m)}) : (A_i \Rightarrow A_i') \to (B \otimes \underline{X}_{m<i} \otimes \underline{X}_{m>i} \Rightarrow B' \otimes \underline{X}'_{m<i} \otimes \underline{X}'_{m>i})$$

Theorem (Compositionality)

For any symmetric monoidal category C *then* pslot[C] *is a symmetric polycategory, furthermore,* C *is enriched in* pslot[C].

Theorem (Reconstruction)

Let QC *and* U *be the categories of finite dimensional channels and unitaries respectively, then:*

$$\mathsf{QS} \cong_{\mathsf{poly}} \mathsf{pslot}[\mathsf{QC}] \qquad \textit{and} \qquad \mathsf{Su} \cong_{\mathsf{poly}} \mathsf{pslot}[\mathsf{U}]$$

Summary

The **pslot**[C] construction:

- Always returns a symmetric polycategory
- Which enriches the monoidal structure of C
- Which recovers the superchannels from the quantum channels
- Which recovers the superunitaries from the unitaries

Cool, so are we done? …

Categorical Weaknesses/Future Avenues ...

Weaknesses form the aesthetic point of view:

- Not obviously functorial
- Not obviously universal
- Not obviously a bare-minimum
- Relies heavily on symmetries
- Representability ($\otimes, \boxtimes$)?
- Structure preserving maps should be changes of base for enrichment
- Should we expect even stronger compositional features?

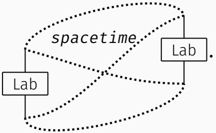
Need a general meta-theory for comparison of approaches

[M. Wilson and G. Chiribella, arXiv 2022][G. S. H. Cruttwell, PhD Thesis]

New Avenues 1

We have working supermap definitions for OPTs and arbitrary Hilbert spaces

- Characterise and compare with concrete approaches in ∞-dimensional setting
- Post-quantum causal structures?



[G. Chiribella, G. M. D'Ariano, and P. Perinotti, PRA 2009] [G. Chiribella, A. Toigo, V. Umanità, Open Systems and Information Dynamics 2013]

[F. Giacomini, E. Castro-Ruiz, Č. Brukner, New J Phys 2016]

New Avenues 2

Beyond quantum supermaps is the general framework of higher order quantum theory

- Constructions of higher-order quantum theory allow for super-super-super...maps
- Can we reconstruct all layers from the **pslot**-construction?

$$(A_1 \Rightarrow A_1{}') \Rightarrow (A_2 \Rightarrow A_2{}') \longrightarrow (B \Rightarrow B')$$

In short, can we generalize construction of higher-order causal categories to arbitrary symmetric monoidal categories?

[A. Bisio and P. Perinotti, Royal Society A 2019] [L. Apadula, A. Bisio, P. Perinotti, arXiv 2022] [A. Kissinger, S. Uijlen, LMiCS 2019] [W. Simmons, A. Kissinger, arXiv 2022]

Thank-You for Listening!

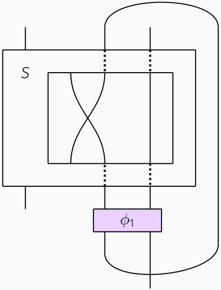
- Quantum Supermaps are Characterized by Locality

https://arxiv.org/abs/2205.09844

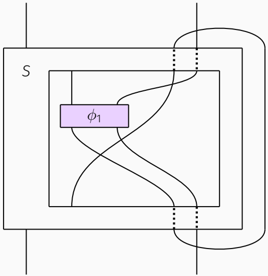
- Free Polycategories for Unitary Supermaps of Arbitrary Dimension

https://arxiv.org/abs/2207.09180

Sketch Proof



Sketch Proof



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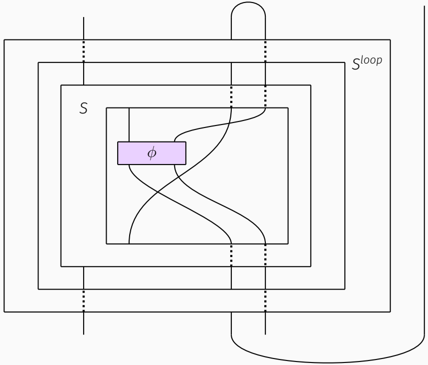

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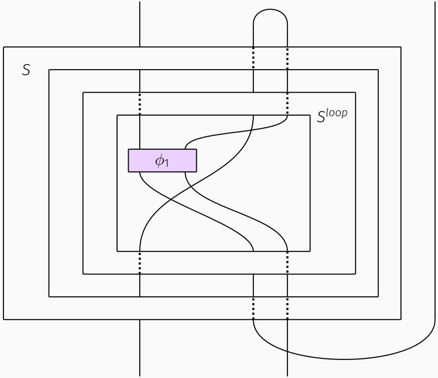

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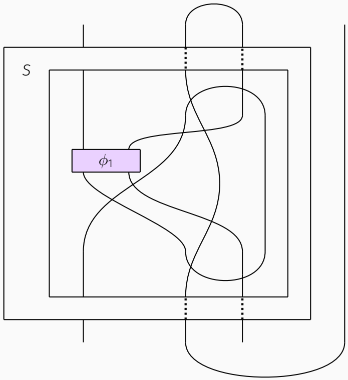

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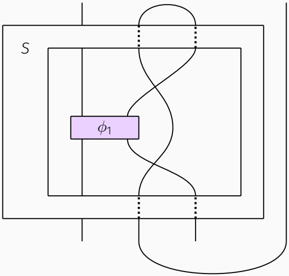

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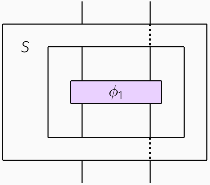
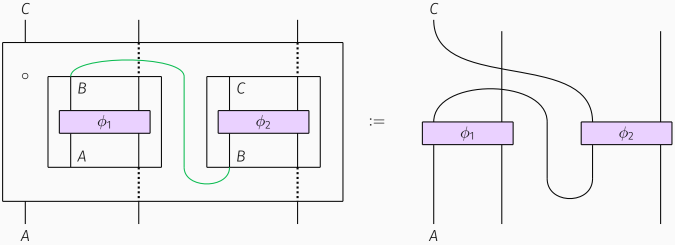


Enriched Structure for Supermaps

Locally-applicable transformations enrich the category on which they act



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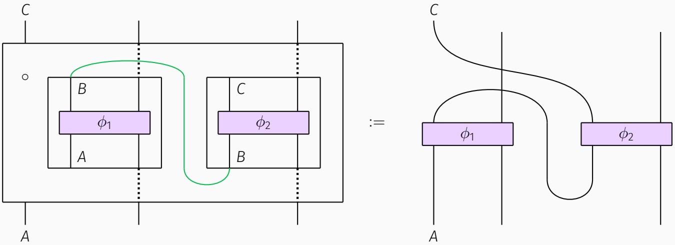
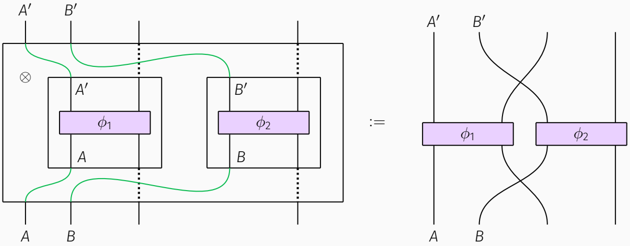


Formally have constructed a **lot[C]**-category **C**

Locally-applicable transformations enrich the category on which they act



Formally have constructed a **lot[C]**-monoidal category **C**